

REDUCED CLASSICAL FIELD THEORIES. k -COSYMPLECTIC FORMALISM ON LIE ALGEBROIDS

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ABSTRACT. In this paper we introduce a geometric description of Lagrangian and Hamiltonian classical field theories on Lie algebroids in the framework of k -cosymplectic geometry. We discuss the relation between Lagrangian and Hamiltonian descriptions through a convenient notion of Legendre transformation. The theory is a natural generalization of the standard one; in addition, other interesting examples are studied, mainly on reduction of classical field theories.

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1. INTRODUCTION

The k -cosymplectic formalism is one of the simplest geometric framework for describing many interesting cases of first-order classical field theories. It is a generalization to field theories of the standard cosymplectic formalism for non-autonomous mechanics and it is adequate for describing field theories with Lagrangians or Hamiltonians function explicitly depending on coordinates in the basis or the set of parameters. The foundation of the k -cosymplectic formalism is the k -cosymplectic manifolds [30, 31].

Historically, it is based on the so-called *polysymplectic formalism* developed by Günther [19], who introduced the *polysymplectic manifolds*. A refinement of this concept allows us to define *k -symplectic manifolds* [2, 3], which are polysymplectic manifolds admitting *Darboux-type coordinates* [27, 28]. (Other

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different polysymplectic formalisms for describing field theories have been also proposed [15, 16, 21, 41, 46, 50]).

Sometimes, the Lagrangian and Hamiltonian functions are not defined on a k -cosymplectic manifold, for instance, in the reduction theory, where the reduced “phase spaces” are not, in general, k -cosymplectic manifolds, even when the original phase space is a k -cosymplectic manifold. For instance, in this paper, we will see that when we consider reduction by symmetry of Lagrangian field theories, we obtain a reduced Lagrangian which can not be described using the standard k -cosymplectic theory. In Mechanics this problem is solved using Lie algebroids instead of tangent and cotangent bundles (see [25, 55]).

The goal of this paper is to develop an extension of k -cosymplectic field theories to Lie algebroids, such that, in the particular case $k = 1$ we obtain the traditional mechanics on Lie algebroids and when the Lie algebroid is the tangent bundle we derive the classical k -cosymplectic formalism. Classical field theories on Lie algebroids have already been studied in the literature. For instance, the multisymplectic formalism on Lie algebroids was presented in [39, 40], the k -symplectic formalism on Lie algebroids was studied in [24]. In [53] a geometric framework for discrete field theories on Lie groupoids has been discussed.

The organization of the paper is as follows. In section 2 we summarize some aspects of the reduction on principal bundles developed by M. Castrillón *et al.* in [6] and [7], the *covariant Lagrangian reduction*. This approach gives us examples of field theories on reduced “phase spaces” which are not, in general, k -cosymplectic manifolds. Here we observe that is necessary to develop a theory more general than the k -cosymplectic formalism for field theory. In section 3 we recall some basic elements from the k -cosymplectic approach to first order classical field theories. In section 4 we remember some basic facts about Lie algebroids and the differential geometric aspects associated to them. In this section we also describe a particular example of Lie algebroid, called the *prolongation of a Lie algebroid over a fibration*. This Lie algebroid will be necessary for the further developments. In section 5 the k -cosymplectic formalism is extended to the setting of Lie algebroids. The subsection 5.1 describe the Lagrangian approach and the subsection 5.2 describe the Hamiltonian approach. These formalisms are developed in an analogous way to the standard k -cosymplectic Lagrangian and Hamiltonian formalisms. We finish this section defining the Legendre transformation on the context of Lie algebroids and we establish the equivalence between both formalism, Lagrangian and Hamiltonian, when the Lagrangian function is hyperregular. In section 6 we show some examples where the theory can be applied.

All manifolds and maps are C^∞ . Sum over crossed repeated indices is understood. Along this paper one k -tuple of elements will be denoted by a bold symbol.

2. MOTIVATING EXAMPLE: PRINCIPAL BUNDLE REDUCTION, COVARIANT EULER-POINCARÉ EQUATIONS.

Reduction by symmetry of Lagrangian field theories is useful for the implementation of many diverse mathematical models from geometric mechanics. One of the main approaches has been developed by M. Castrillón-López *et al.* in [6] and [7] and it is referred as *covariant Lagrangian reduction*.

The papers on covariant Lagrangian reduction, [7], dealt with the extension of classical Euler-Poincaré reduction of variational principles to the field theoretic context, the idea of this paper is the following: a field theory was formulated on a principal bundle and was reduced by the structure group. This process can be summarized as follows. We begin with a right principal bundle $\pi: P \rightarrow M$ with structure group G . The group G naturally acts on $J^1 P$ by $(j_x^1 s) \cdot g \mapsto j_x^1(R_g \circ s)$, for any $j_x^1 s \in J^1 P$ and $g \in G$. One considers a Lagrangian $L: J^1 P \rightarrow \mathbb{R}$, invariant under the natural action induced by the structure group G . The reduced variational problem now takes place on $\mathcal{C}(P) = (J^1 P)/G$, the bundle of connections, for more details see [7].

Now we consider the following particular case: $P = \mathbb{R}^k \times G$ and $M = \mathbb{R}^k$, that is, the trivial bundle $\mathbb{R}^k \times G \rightarrow \mathbb{R}^k$, in this case, $J^1 P$ can be identified with $\mathbb{R}^k \times T_k^1 G$, where, $T_k^1 G$ is the tangent bundle of k^1 -velocities of G , that is, the Whitney sum of k copies of TG (see section 3).

Let $L: J^1 P \equiv \mathbb{R}^k \times T_k^1 G \rightarrow \mathbb{R}$ be a Lagrangian invariant under the natural action of G on $\mathbb{R}^k \times T_k^1 G$. In this case, we make the identifications

$$J^1(\mathbb{R}^k \times G)/G \cong (\mathbb{R}^k \times T_k^1 G)/G \cong (\mathbb{R}^k \times G \times \mathfrak{g} \times \dots \times \mathfrak{g})/G \cong \mathbb{R}^k \times \mathfrak{g} \times \dots \times \mathfrak{g}$$

and then the reduced Lagrangian is a function $l: (J^1 P)/G \equiv (\mathbb{R}^k \times T_k^1 G)/G \cong \mathbb{R}^k \times (\mathfrak{g} \oplus \dots \oplus \mathfrak{g}) \rightarrow \mathbb{R}$.

Let us observe that in this simple example of covariant Lagrangian reduction, the reduced Lagrangian is not defined on a k -cosymplectic manifold.

In this paper we will study classical field theories on Lie algebroids using the k -cosymplectic approach. In this setting the above example can be solved. Furthermore, we will develop a framework that:

- (1) Reduces the classical k -cosymplectic field theories, [30, 31] to particular cases.
- (2) Reduces mechanics on Lie algebroids, see for instance [9, 37], to particular cases.

3. GEOMETRIC PRELIMINARIES

In this section we recall some basic elements from the k -cosymplectic approach to classical field theories [30, 31].

3.1. The manifold $\mathbb{R}^k \times (T_k^1)^*Q$. Let Q be an n -dimensional differentiable manifold and $\pi_Q : T^*Q \rightarrow Q$ its cotangent bundle. We denote by $(T_k^1)^*Q$ the Whitney sum $T^*Q \oplus \dots \oplus T^*Q$ of k copies of T^*Q .

$(T_k^1)^*Q$ can be identified with the manifold $J^1(Q, \mathbb{R}^k)_0$ of k^1 -covelocities of Q , that is, 1-jets of maps $\sigma : Q \rightarrow \mathbb{R}^k$ with target at $\mathbf{0} \in \mathbb{R}^k$, say

$$\begin{aligned} J^1(Q, \mathbb{R}^k)_0 &\equiv T^*Q \oplus \dots \oplus T^*Q \\ j_q^1 \sigma &\equiv (d\sigma^1(q), \dots, d\sigma^k(q)), \end{aligned}$$

where $\sigma^A = pr_A \circ \sigma : Q \rightarrow \mathbb{R}$ is the A th component of σ and $pr_A : \mathbb{R}^k \rightarrow \mathbb{R}$ are the canonical projections, $1 \leq A \leq k$. For this reason, $(T_k^1)^*Q$ is also called *the bundle of k^1 -covelocities of the manifold Q* .

The manifold $J^1\hat{\pi}_Q$ of 1-jets of sections of the trivial bundle $\hat{\pi}_Q : \mathbb{R}^k \times Q \rightarrow Q$ is diffeomorphic to $\mathbb{R}^k \times (T_k^1)^*Q$, via the diffeomorphism given by

$$\begin{aligned} J^1\hat{\pi}_Q &\longrightarrow \mathbb{R}^k \times (T_k^1)^*Q \\ j_q^1 \phi &= j_q^1(\phi_{\mathbb{R}^k}, Id_Q) \longmapsto (\phi_{\mathbb{R}^k}(q), \alpha_q^1, \dots, \alpha_q^k), \end{aligned}$$

where $\phi_{\mathbb{R}^k} : Q \xrightarrow{\phi} \mathbb{R}^k \times Q \xrightarrow{\hat{\pi}_{\mathbb{R}^k}} \mathbb{R}^k$, $1 \leq A \leq k$, $(\phi_{\mathbb{R}^k})^A : Q \xrightarrow{\phi_{\mathbb{R}^k}} \mathbb{R}^k \xrightarrow{pr_A} \mathbb{R}$ and $\alpha_q^A = d(\phi_{\mathbb{R}^k})^A(q)$.

Throughout all the paper we use the following notation for the canonical projections

$$\begin{array}{ccc} \mathbb{R}^k \times (T_k^1)^*Q & \xrightarrow{(\hat{\pi}_Q)_1, 0} & \mathbb{R}^k \times Q \\ & \searrow (\hat{\pi}_Q)_1 & \downarrow \hat{\pi}_Q \\ & & Q \end{array}$$

where

$$\hat{\pi}_Q(\mathbf{t}, q) = q, \quad (\hat{\pi}_Q)_{1,0}(\mathbf{t}, \alpha_q^1, \dots, \alpha_q^k) = (\mathbf{t}, q), \quad (\hat{\pi}_Q)_1(\mathbf{t}, \alpha_q^1, \dots, \alpha_q^k) = q,$$

with $\mathbf{t} \in \mathbb{R}^k$, $q \in Q$ and $(\alpha_q^1, \dots, \alpha_q^k) \in (T_k^1)^*Q$.

If (q^i) are local coordinates on $U \subseteq Q$, then the induced local coordinates (q^i, p_i) , $1 \leq i \leq n$, on $(\pi_Q)^{-1}(U) = T^*U \subset T^*Q$, are expressed by

$$q^i(\alpha_q) = q^i(q), \quad p_i(\alpha_q) = \alpha_q \left(\frac{\partial}{\partial q^i} \Big|_q \right),$$

and the induced local coordinates (t^A, q^i, p_i^A) , $1 \leq i \leq n$, $1 \leq A \leq k$, on $[(\hat{\pi}_Q)_1]^{-1}(U) = \mathbb{R}^k \times (T_k^1)^*U$ are given by

$$t^A(\mathbf{t}, \alpha_q^1, \dots, \alpha_q^k) = t^A, \quad q^i(\mathbf{t}, \alpha_q^1, \dots, \alpha_q^k) = q^i(q), \quad p_i^A(\mathbf{t}, \alpha_q^1, \dots, \alpha_q^k) = \alpha_q^A \left(\frac{\partial}{\partial q^i} \Big|_q \right).$$

On $\mathbb{R}^k \times (T_k^1)^*Q$, we consider the differential forms

$$\eta^A \equiv dt^A = (\pi_1^A)^*dt^A, \quad \theta^A = (\pi_2^A)^*\theta, \quad \omega^A = (\pi_2^A)^*\omega,$$

$\pi_1^A : \mathbb{R}^k \times (T_k^1)^*Q \rightarrow \mathbb{R}$, $\pi_2^A : \mathbb{R}^k \times (T_k^1)^*Q \rightarrow T^*Q$ being the canonical projections defined by

$$\pi_1^A(\mathbf{t}, \alpha_q^1, \dots, \alpha_q^k) = t^A, \quad \pi_2^A(\mathbf{t}, \alpha_q^1, \dots, \alpha_q^k) = \alpha_q^A,$$

where $\omega = -d\theta = dq^i \wedge dp_i$ is the canonical symplectic form on T^*Q and $\theta = p_i dq^i$ is the Liouville 1-form on T^*Q . Obviously $\omega^A = -d\theta^A$, $1 \leq A \leq k$.

In local coordinates we have

$$(3.1) \quad \theta^A = p_i^A dq^i, \quad \omega^A = dq^i \wedge dp_i^A.$$

Moreover, let

$$V^* = \ker((\hat{\pi}_Q)_{1,0})_* = \left\langle \frac{\partial}{\partial p_i^1}, \dots, \frac{\partial}{\partial p_i^k} \right\rangle_{i=1,\dots,n},$$

be the vertical distribution of the bundle $(\hat{\pi}_Q)_{1,0} : \mathbb{R}^k \times (T_k^1)^* Q \rightarrow \mathbb{R}^k \times Q$.

A simple inspection of the expressions in local coordinates (3.1) shows that the forms η^A and ω^A are closed, and the following relations hold

- (1) $\eta^1 \wedge \dots \wedge \eta^k \neq 0$, $(\eta^A)|_{V^*} = 0$, $(\omega^A)|_{V^* \times V^*} = 0$,
- (2) $(\cap_{A=1}^k \ker \eta^A) \cap (\cap_{A=1}^k \ker \omega^A) = \{0\}$, $\dim(\cap_{A=1}^k \ker \omega^A) = k$,

From the above geometrical model, the following definition is introduced in [30]:

Definition 3.1. *Let M be a differentiable manifold of dimension $k(n+1)+n$. A k -cosymplectic structure on M is a family $(\eta^A, \omega^A, V; 1 \leq A \leq k)$, where each η^A is a closed 1-form, each ω^A is a closed 2-form and V is an integrable nk -dimensional distribution on M , satisfying (i) and (ii).*

M is said to be an k -cosymplectic manifold.

The following theorem has been proved in [30].

Theorem 3.2. (Darboux Theorem): *If M is an k -cosymplectic manifold, then around each point of M there exist local coordinates (t^A, q^i, p_i^A) such that*

$$\eta^A = dt^A, \quad \omega^A = dq^i \wedge dp_i^A, \quad V = \left\langle \frac{\partial}{\partial p_i^1}, \dots, \frac{\partial}{\partial p_i^k} \right\rangle_{i=1,\dots,n}.$$

In consequence, the canonical model for these geometrical structures is $(\mathbb{R}^k \times (T_k^1)^* Q, \eta^A, \omega^A, V^*)$. See, for instance [30, 31, 44]

3.2. The manifold $\mathbb{R}^k \times T_k^1 Q$. Let Q be an n -dimensional manifold and $\tau_Q : TQ \rightarrow Q$ its tangent bundle. We denote by $T_k^1 Q$ the Whitney sum $TQ \oplus \mathbb{R}^k \oplus \dots \oplus TQ$ of k copies of TQ , with projection $\tau_Q^k : T_k^1 Q \rightarrow Q$, $\tau_Q^k(v_{1q}, \dots, v_{kq}) = q$, where $v_{Aq} \in T_q Q$, $A = 1, \dots, k$. $T_k^1 Q$ can be identified with the manifold $J_0^1(\mathbb{R}^k, Q)$ of k^1 -velocities of Q , that is 1-jets of maps $\sigma : \mathbb{R}^k \rightarrow Q$ with the source at $\mathbf{0} \in \mathbb{R}^k$, say

$$\begin{aligned} J_0^1(\mathbb{R}^k, Q) &\equiv TQ \oplus \mathbb{R}^k \oplus \dots \oplus TQ \\ j_{0,q}^1 \sigma &\equiv (v_{1q}, \dots, v_{kq}) \end{aligned}$$

where $q = \sigma(\mathbf{0})$ and $v_{Aq} = T_{\mathbf{0}} \sigma \left(\frac{\partial}{\partial t^A} \Big|_{\mathbf{t}=\mathbf{0}} \right)$, (t^1, \dots, t^k) being the standard coordinates on \mathbb{R}^k . $T_k^1 Q$ is called the *tangent bundle of k^1 -velocities of Q* (see [42]).

The manifold $J^1 \hat{\pi}_{\mathbb{R}^k}$ of 1-jets of sections of the trivial bundle $\hat{\pi}_{\mathbb{R}^k} : \mathbb{R}^k \times Q \rightarrow \mathbb{R}^k$ is diffeomorphic to $\mathbb{R}^k \times T_k^1 Q$, via the diffeomorphism given by

$$\begin{aligned} J^1 \hat{\pi}_{\mathbb{R}^k} &\rightarrow \mathbb{R}^k \times T_k^1 Q \\ j_t^1 \phi &= j_t^1(Id_{\mathbb{R}^k}, \phi_Q) \rightarrow (t, v_1, \dots, v_k) \end{aligned}$$

where $\phi_Q : \mathbb{R}^k \xrightarrow{\phi} \mathbb{R}^k \times Q \xrightarrow{\hat{\pi}_Q} Q$, and

$$v_A = T_{\mathbf{t}} \phi_Q \left(\frac{\partial}{\partial t^A} \Big|_{\mathbf{t}} \right), \quad 1 \leq A \leq k.$$

Let $p_Q : \mathbb{R}^k \times T_k^1 Q \rightarrow Q$ be the canonical projection. If (q^i) are local coordinates on $U \subseteq Q$, then the induced local coordinates (t^A, q^i, v_A^i) on $p_Q^{-1}(U) = \mathbb{R}^k \times T_k^1 U$ are expressed by

$$t^A(\mathbf{t}, v_{1q}, \dots, v_{kq}) = t^A; \quad q^i(\mathbf{t}, v_{1q}, \dots, v_{kq}) = q^i(q); \quad v_A^i(\mathbf{t}, v_{1q}, \dots, v_{kq}) = \langle dq^i, v_A \rangle,$$

where $1 \leq i \leq n$, $1 \leq A \leq k$.

Throughout the paper we use the following notation for the canonical projections

$$\begin{array}{ccc} \mathbb{R}^k \times T_k^1 Q & \xrightarrow{(\hat{\pi}_{\mathbb{R}^k})_{1,0}} & \mathbb{R}^k \times Q \\ & \searrow (\hat{\pi}_{\mathbb{R}^k})_1 & \downarrow \hat{\pi}_{\mathbb{R}^k} \\ & & \mathbb{R}^k \end{array}$$

where, for $\mathbf{t} \in \mathbb{R}^k$, $q \in Q$ and $(v_{1q}, \dots, v_{kq}) \in T_k^1 Q$,

$$\hat{\pi}_{\mathbb{R}^k}(\mathbf{t}, q) = \mathbf{t}, \quad (\hat{\pi}_{\mathbb{R}^k})_{1,0}(\mathbf{t}, v_{1q}, \dots, v_{kq}) = (\mathbf{t}, q), \quad (\hat{\pi}_{\mathbb{R}^k})_1(\mathbf{t}, v_{1q}, \dots, v_{kq}) = \mathbf{t}.$$

3.3. k-vector fields and integral sections.

Let M be an arbitrary manifold.

Definition 3.3. A section $\mathbf{X} : M \longrightarrow T_k^1 M$ of the projection τ_M^k will be called a k -vector field on M .

To give a k -vector field \mathbf{X} is equivalent to give a family of k vector fields X_1, \dots, X_k . Hence in the sequel we will indistinctly write $\mathbf{X} = (X_1, \dots, X_k)$.

Definition 3.4. An integral section of the k -vector field $\mathbf{X} = (X_1, \dots, X_k)$, passing through a point $x \in M$, is a map $\psi : U_0 \subset \mathbb{R}^k \rightarrow M$, defined on some neighborhood U_0 of $\mathbf{0} \in \mathbb{R}^k$, such that $\psi(\mathbf{0}) = x$, and

$$(3.2) \quad \psi_*(\mathbf{t}) \left(\frac{\partial}{\partial t^A} \Big|_{\mathbf{t}} \right) = X_A(\psi(\mathbf{t})), \quad \text{for every } \mathbf{t} \in U_0, 1 \leq A \leq k,$$

or, equivalently, $\psi(\mathbf{0}) = x$ and ψ satisfies $\mathbf{X} \circ \psi = \psi^{(1)}$, where $\psi^{(1)}$ is the first prolongation of ψ to $T_k^1 M$, defined by

$$\begin{aligned} \psi^{(1)} : U_0 \subset \mathbb{R}^k &\longrightarrow T_k^1 M \\ \mathbf{t} &\longrightarrow \psi^{(1)}(\mathbf{t}) = j_{\mathbf{0}}^1 \psi_{\mathbf{t}} \equiv \left(\psi_*(\mathbf{t}) \left(\frac{\partial}{\partial t^1} \Big|_{\mathbf{t}} \right), \dots, \psi_*(\mathbf{t}) \left(\frac{\partial}{\partial t^k} \Big|_{\mathbf{t}} \right) \right), \end{aligned}$$

where $\psi_{\mathbf{t}}(\mathbf{s}) = \psi(\mathbf{t} + \mathbf{s})$.

A k -vector field $\mathbf{X} = (X_1, \dots, X_k)$ on M is said to be integrable if there is an integral section passes through every point of M .

Remark 3.5. In the k -cosymplectic formalism, the solutions of Euler-Lagrange's field equations are the integral sections of k -vector fields. In the case $k = 1$, this definition coincides with the classical definition of the integral curve of a vector field. \diamond

4. LIE ALGEBROIDS

In this section we present some basic facts about Lie algebroids, including features of the associated differential calculus and results on Lie algebroid morphisms that will be necessary. For further information on groupoids and Lie algebroids, and their roles in differential geometry, see [4, 20, 32, 33].

4.1. Lie algebroid: definition. Let E be a vector bundle of rank m over a manifold Q of dimension n , and let $\tau : E \rightarrow Q$ be the vector bundle projection. Denote by $\text{Sec}(E)$ the $C^\infty(Q)$ -module of sections of τ . A *Lie algebroid structure* $(\llbracket \cdot, \cdot \rrbracket_E, \rho_E)$ on E is a Lie bracket $\llbracket \cdot, \cdot \rrbracket_E : \text{Sec}(E) \times \text{Sec}(E) \rightarrow \text{Sec}(E)$ on the space $\text{Sec}(E)$, together with an *anchor map* $\rho_E : E \rightarrow TQ$ and its, identically denoted, induced $C^\infty(Q)$ -module homomorphism $\rho_E : \text{Sec}(E) \rightarrow \mathfrak{X}(Q)$ such that the *compatibility condition*

$$\llbracket \sigma_1, f\sigma_2 \rrbracket_E = f\llbracket \sigma_1, \sigma_2 \rrbracket_E + (\rho_E(\sigma_1)f)\sigma_2,$$

holds for any smooth functions f on Q and sections σ_1, σ_2 of E (here $\rho_E(\sigma_1)$ is the vector field on Q given by $\rho_E(\sigma_1)(q) = \rho_E(\sigma_1(q))$). The triple $(E, \llbracket \cdot, \cdot \rrbracket_E, \rho_E)$ is called a *Lie algebroid over Q* . From the compatibility condition and the Jacobi identity, it follows that $\rho_E : \text{Sec}(E) \rightarrow \mathfrak{X}(Q)$ is a homomorphism between the Lie algebras $(\text{Sec}(E), \llbracket \cdot, \cdot \rrbracket_E)$ and $(\mathfrak{X}(Q), [\cdot, \cdot])$. The following are examples of Lie algebroids.

- (1) **Real Lie algebras of finite dimension.** Any real Lie algebra of finite dimension is a Lie algebroid over a single point.
- (2) **The tangent bundle.** If TQ is the tangent bundle of a manifold Q , then, the triple $(TQ, [\cdot, \cdot], id_{TQ})$ is a Lie algebroid over Q , where $id_{TQ} : TQ \rightarrow TQ$ is the identity map.
- (3) Another important example of a Lie algebroid may be constructed as follows. Let $\pi : P \rightarrow Q$ be a principal bundle with structural group G . Denote by $\Phi : G \times P \rightarrow P$ the free action of G on P and by $T\Phi : G \times TP \rightarrow TP$ the tangent action of G on TP . Then the sections of the quotient vector bundle $\tau_{P|G} : TP/G \rightarrow Q = P/G$ may be identified with the vector fields on P which are invariant under the action Φ . Since every G -invariant vector field on P is π -projectable and the standard Lie bracket on vector fields is closed with respect to G -invariant vector fields, we can define a Lie algebroid structure on TP/G . This Lie algebroid over Q is called **the Atiyah (gauge) algebroid associated with the principal bundle** $\pi : P \rightarrow Q$ [25, 32].

Throughout this paper, the role played by a Lie algebroid is the same as the tangent bundle of Q . In this way, one regards an element e of E as a generalized velocity, and the actual velocity v is obtained when we apply the anchor map to e , i.e. $v = \rho_E(e)$.

Let $(q^i)_{i=1}^n$ be local coordinates on a neighborhood U of Q and $\{e_\alpha\}_{1 \leq \alpha \leq m}$ a local basis of sections of τ . Given $e \in E$ such that $\tau(e) = q$, we can write $e = y^\alpha(e)e_\alpha(q) \in E_q$, i.e. each section σ is given locally by $\sigma|_U = y^\alpha e_\alpha$ and the coordinates of e are $(q^i(e), y^\alpha(e))$. A Lie algebroid structure on Q is locally determined as a set of local *structure functions* ρ_α^i , $\mathcal{C}_{\alpha\beta}^\gamma$ on Q that are defined by

$$(4.1) \quad \rho_E(e_\alpha) = \rho_\alpha^i \frac{\partial}{\partial q^i}, \quad [e_\alpha, e_\beta]_E = \mathcal{C}_{\alpha\beta}^\gamma e_\gamma$$

and satisfy the relations

$$(4.2) \quad \sum_{cyclic(\alpha, \beta, \gamma)} \left(\rho_\alpha^i \frac{\partial \mathcal{C}_{\beta\gamma}^\nu}{\partial q^i} + \mathcal{C}_{\alpha\mu}^\nu \mathcal{C}_{\beta\gamma}^\mu \right) = 0, \quad \rho_\alpha^j \frac{\partial \rho_\beta^i}{\partial q^j} - \rho_\beta^j \frac{\partial \rho_\alpha^i}{\partial q^j} = \rho_\gamma^i \mathcal{C}_{\alpha\beta}^\gamma.$$

These relations, which are a consequence of the compatibility condition and Jacobi's identity, are usually called *the structure equations* of the Lie algebroid E .

4.2. Exterior differential. A Lie algebroid structure on E allows us to define *the exterior differential* of E , $d^E : \text{Sec}(\bigwedge^l E^*) \rightarrow \text{Sec}(\bigwedge^{l+1} E^*)$, as follows:

$$(4.3) \quad \begin{aligned} d^E \mu(\sigma_1, \dots, \sigma_{l+1}) &= \sum_{i=1}^{l+1} (-1)^{i+1} \rho_E(\sigma_i) \mu(\sigma_1, \dots, \hat{\sigma}_i, \dots, \sigma_{l+1}) \\ &+ \sum_{i < j} (-1)^{i+j} \mu([\sigma_i, \sigma_j]_E, \sigma_1, \dots, \hat{\sigma}_i, \dots, \hat{\sigma}_j, \dots, \sigma_{l+1}), \end{aligned}$$

for $\mu \in \text{Sec}(\bigwedge^l E^*)$ and $\sigma_1, \dots, \sigma_{l+1} \in \text{Sec}(E)$. It follows that d^E is a cohomology operator, that is, $(d^E)^2 = 0$.

In particular, if $f : Q \rightarrow \mathbb{R}$ is a smooth real function then $d^E f(\sigma) = \rho_E(\sigma)f$, for $\sigma \in \text{Sec}(E)$. Locally, the exterior differential is determined by

$$(4.4) \quad d^E q^i = \rho_\alpha^i e^\alpha \quad \text{and} \quad d^E e^\gamma = -\frac{1}{2} \mathcal{C}_{\alpha\beta}^\gamma e^\alpha \wedge e^\beta,$$

where $\{e^\alpha\}$ is the dual basis of $\{e_\alpha\}$.

The usual Cartan calculus extends to the case of Lie algebroids: for every section σ of E we have a derivation ι_σ (contraction) of degree -1 and a derivation $\mathcal{L}_\sigma = \iota_\sigma \circ d + d \circ \iota_\sigma$ (the Lie derivative) of degree 0 ; for more details, see [32, 33].

4.3. Morphisms. Let $(E, [\cdot, \cdot]_E, \rho_E)$ and $(E', [\cdot, \cdot]_{E'}, \rho_{E'})$ be two Lie algebroids over Q and Q' respectively, and suppose that $\Phi = (\overline{\Phi}, \underline{\Phi})$ is a vector bundle map, that is $\overline{\Phi} : E \rightarrow E'$ is a fiberwise linear map over $\underline{\Phi} : Q \rightarrow Q'$. The pair $(\overline{\Phi}, \underline{\Phi})$ is said to be a *Lie algebroid morphism* if

$$(4.5) \quad d^E(\Phi^* \sigma') = \Phi^*(d^{E'} \sigma'), \quad \text{for all } \sigma' \in \text{Sec}(\bigwedge^l (E')^*) \text{ and for all } l.$$

Here $\Phi^* \sigma'$ is the section of the vector bundle $\bigwedge^l E^* \rightarrow Q$ defined (for $l > 0$) by

$$(4.6) \quad (\Phi^* \sigma')_q(e_1, \dots, e_l) = \sigma'_{\underline{\Phi}(q)}(\overline{\Phi}(e_1), \dots, \overline{\Phi}(e_l)),$$

for $q \in Q$ and $e_1, \dots, e_l \in E_q$. In particular, when $Q = Q'$ and $\underline{\Phi} = id_Q$ then (4.5) holds if and only if

$$[\overline{\Phi} \circ \sigma_1, \overline{\Phi} \circ \sigma_2]_{E'} = \overline{\Phi}[\sigma_1, \sigma_2]_E, \quad \rho_{E'}(\overline{\Phi} \circ \sigma) = \rho_E(\sigma), \quad \text{for } \sigma, \sigma_1, \sigma_2 \in \text{Sec}(E).$$

Let (q^i) be a local coordinate system on Q and (\bar{q}^i) a local coordinate system on Q' . Let $\{e_\alpha\}$ and $\{\bar{e}_\alpha\}$ be local base of sections of E and E' , respectively, and $\{e^\alpha\}$ and $\{\bar{e}^\alpha\}$ their respective dual base. The vector bundle map Φ is determined by the relations $\Phi^* \bar{q}^i = \phi^i(q)$ and $\Phi^* \bar{e}^\alpha = \phi_\beta^\alpha e^\beta$ for certain local functions ϕ^i and ϕ_β^α on Q . In this coordinate system $\Phi = (\overline{\Phi}, \underline{\Phi})$ is a Lie algebroid morphism if and only if

$$(4.7) \quad (\rho_E)_\alpha^j \frac{\partial \phi^i}{\partial q^j} = (\rho_{E'})_{\bar{\beta}}^i \phi_\alpha^{\bar{\beta}}, \quad \phi_\gamma^{\bar{\beta}} \mathcal{C}_{\alpha\delta}^\gamma = \left((\rho_E)_\alpha^i \frac{\partial \phi_{\bar{\beta}}^{\bar{\beta}}}{\partial q^i} - (\rho_E)_\delta^i \frac{\partial \phi_\alpha^{\bar{\beta}}}{\partial q^i} \right) + \bar{\mathcal{C}}_{\bar{\theta}\bar{\sigma}}^{\bar{\beta}} \phi_\alpha^{\bar{\theta}} \phi_{\bar{\sigma}}^{\bar{\beta}},$$

where the $(\rho_E)_\alpha^i$, $\mathcal{C}_{\alpha\beta}^\gamma$ are the structure functions on E and the $(\rho_{E'})_{\bar{\alpha}}^i$, $\bar{\mathcal{C}}_{\bar{\theta}\bar{\sigma}}^{\bar{\beta}}$ are the structure functions on E' .

For more definitions and properties about the concept of Lie algebroid morphism, see for instance [9, 20, 39, 40].

4.4. The prolongation of a Lie algebroid over a fibration. In this subsection we recall a particular kind of Lie algebroid that will be used later (see [9, 20, 25, 37], for more details).

If $(E, [\cdot, \cdot]_E, \rho_E)$ is a Lie algebroid over a manifold Q and $\pi : P \rightarrow Q$ is a fibration, then

$$\tilde{\tau}_P : \mathcal{T}^E P = \bigcup_{p \in P} \mathcal{T}_p^E P \rightarrow P,$$

where

$$\mathcal{T}_p^E P = \{(e, v_p) \in E_{\pi(p)} \times T_p P \mid \rho_E(e) = T_p \pi(v_p)\}$$

is a Lie algebroid called the prolongation of the Lie algebroid $(E, [\cdot, \cdot]_E, \rho_E)$ (see for instance [20, 25]). The anchor map of this Lie algebroid is $\rho^\pi : \mathcal{T}^E P \rightarrow TP$, $\rho^\pi(e, v_p) = v_p$. In this paper we consider two particular Lie algebroid prolongations, one with $P = \mathbb{R}^k \times (E \oplus \overset{k}{\dots} \oplus E)$ and the other with $P = \mathbb{R}^k \times (E^* \oplus \overset{k}{\dots} \oplus E^*)$ (for more details see [9, 20, 25, 37]).

If (q^i, u^ℓ) are local coordinates on P and $\{e_\alpha\}$ is a local basis of sections of E , then a local basis of $\tilde{\tau}_P : \mathcal{T}^E P \rightarrow P$ is given by the family $\{\mathcal{X}_\alpha, \mathcal{V}_\ell\}$ where

$$(4.8) \quad \mathcal{X}_\alpha(p) = (e_\alpha(\pi(p)); \rho_\alpha^i(\pi(p)) \frac{\partial}{\partial q^i} \Big|_p) \quad \text{and} \quad \mathcal{V}_\ell(p) = (0_{\pi(p)}; \frac{\partial}{\partial u^\ell} \Big|_p).$$

The Lie bracket of two sections of $\mathcal{T}^E P$ is characterized by the relations

$$(4.9) \quad [\mathcal{X}_\alpha, \mathcal{X}_\beta]^\pi = \mathcal{C}_{\alpha\beta}^\gamma \mathcal{X}_\gamma \quad [\mathcal{X}_\alpha, \mathcal{V}_\ell]^\pi = 0 \quad [\mathcal{V}_\ell, \mathcal{V}_\varphi]^\pi = 0,$$

and the exterior differential is therefore determined by

$$(4.10) \quad \begin{aligned} d^{\mathcal{T}^E P} q^i &= \rho_\alpha^i \mathcal{X}^\alpha, & d^{\mathcal{T}^E P} u^\ell &= \mathcal{V}^\ell \\ d^{\mathcal{T}^E P} \mathcal{X}^\gamma &= -\frac{1}{2} \mathcal{C}_{\alpha\beta}^\gamma \mathcal{X}^\alpha \wedge \mathcal{X}^\beta, & d^{\mathcal{T}^E P} \mathcal{V}^\ell &= 0 \end{aligned}$$

where $\{\mathcal{X}^\alpha, \mathcal{V}^\ell\}$ is the dual basis of $\{\mathcal{X}_\alpha, \mathcal{V}_\ell\}$.

5. CLASSICAL FIELD THEORIES ON LIE ALGEBROIDS: A k -COSYMPLECTIC APPROACH

In this section, the k -cosymplectic formalism for first order classical field theories (see [30, 31]) is extended to the general setting of Lie algebroids. Considering a Lie algebroid E as a generalization of the tangent bundle TQ of Q , we will define the analog of the classical field equations and their solutions, and we study the analogs of the geometric structures of the standard k -cosymplectic formalism. Lagrangian and Hamiltonian formalisms are developed in subsections 5.1 and 5.2, respectively, and it is verified that the standard Lagrangian and Hamiltonian k -cosymplectic formalisms are particular examples of the formalism developed here. Throughout this section we consider a Lie algebroid $(E, [\cdot, \cdot]_E, \rho_E)$ (E for simplicity) on the manifold Q .

5.1. Lagrangian formalism. First, we will introduce some geometric ingredients which are necessary to develop the Lagrangian k -cosymplectic formalism on Lie algebroids.

5.1.1. *The manifold $\mathbb{R}^k \times \overset{k}{\oplus} E$.* The standard k -cosymplectic Lagrangian formalism is developed on the bundle $\mathbb{R}^k \times T_k^1 Q$, where $T_k^1 Q \equiv TQ \oplus \overset{k}{\dots} \oplus TQ$ is the Whitney sum of k copies of TQ . Since we are thinking of a Lie algebroid E as a substitute of the tangent bundle, it is natural to consider

$$\mathbb{R}^k \times \overset{k}{\oplus} E \equiv \mathbb{R}^k \times (E \oplus \overset{k}{\dots} \oplus E),$$

and the projection map $\tilde{p} : \mathbb{R}^k \times \overset{k}{\oplus} E \rightarrow Q$, given by $\tilde{p}(t^1, \dots, t^k, e_{1q}, \dots, e_{kq}) = q$.

Let us observe that the elements of $\mathbb{R}^k \times \overset{k}{\oplus} E$ have the following form:

$$(\mathbf{t}, \mathbf{e}_q) = (t^1, \dots, t^k, e_{1q}, \dots, e_{kq}).$$

If (q^i, y^α) are local coordinates on $\tau^{-1}(U) \subseteq E$, then the induced local coordinates (t^A, q^i, y_A^α) on $\tilde{p}^{-1}(U) \subseteq \mathbb{R}^k \times \overset{k}{\oplus} E$ are given by

$$(5.1) \quad t^A(\mathbf{t}, \mathbf{e}_q) = t^A(\mathbf{t}), \quad q^i(\mathbf{t}, \mathbf{e}_q) = q^i(q), \quad y_A^\alpha(\mathbf{t}, \mathbf{e}_q) = y_A^\alpha(e_{Aq}).$$

5.1.2. *The Lagrangian prolongation.* Consider the prolongation of a Lie algebroid E over the fibration $\tilde{p}: \mathbb{R}^k \times \bigoplus^k E \rightarrow Q$, (see section 4.4),

$$(5.2) \quad \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E) = \{(a_q, v_{(\mathbf{t}, \mathbf{e}_q)}) \in E \times T(\mathbb{R}^k \times \bigoplus^k E) / \rho_E(a_q) = T\tilde{p}(v_{(\mathbf{t}, \mathbf{e}_q)})\},$$

where $(\mathbf{t}, \mathbf{e}_q) \in \mathbb{R}^k \times \bigoplus^k E$. We deduce the following properties (see [9, 25, 37] for standard properties of the prolongation Lie algebroid):

(1) $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E) \subset E \times T(\mathbb{R}^k \times \bigoplus^k E)$, with projection

$$\tilde{\tau}_{\mathbb{R}^k \times \bigoplus^k E}: \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E) \longrightarrow \mathbb{R}^k \times \bigoplus^k E$$

has a Lie algebroid structure $([\![\cdot, \cdot]\!], \tilde{\rho})$, where the anchor map

$$\rho^{\tilde{\rho}}: \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E) \rightarrow T(\mathbb{R}^k \times \bigoplus^k E)$$

is the canonical projection on the second factor. In the sequel, this induced Lie algebroid structure will be called the *Lagrangian prolongation*.

(2) If (t^A, q^i, y_A^α) are local coordinates on $\mathbb{R}^k \times \bigoplus^k E$, then the induced local coordinates on $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)$ are

$$(t^A, q^i, y_A^\alpha, z^\alpha, v_A, w_A^\alpha)_{1 \leq i \leq n, 1 \leq A \leq k, 1 \leq \alpha \leq m}$$

where

$$(5.3) \quad \begin{aligned} t^A(a_q, v_{(\mathbf{t}, \mathbf{e}_q)}) &= t^A(\mathbf{t}), & z^\alpha(a_q, v_{(\mathbf{t}, \mathbf{e}_q)}) &= y^\alpha(a_q), \\ q^i(a_q, v_{(\mathbf{t}, \mathbf{e}_q)}) &= q^i(q), & v_A(a_q, v_{(\mathbf{t}, \mathbf{e}_q)}) &= v_{(\mathbf{t}, \mathbf{e}_q)}(t^A), \\ y_A^\alpha(a_q, v_{(\mathbf{t}, \mathbf{e}_q)}) &= y_A^\alpha(\mathbf{e}_q), & w_A^\alpha(a_q, v_{(\mathbf{t}, \mathbf{e}_q)}) &= v_{(\mathbf{t}, \mathbf{e}_q)}(y_A^\alpha). \end{aligned}$$

(3) The set $\{\mathcal{Y}_A, \mathcal{X}_\alpha, \mathcal{V}_\alpha^A\}$ given by

$$\mathcal{Y}_A, \mathcal{X}_\alpha, \mathcal{V}_\alpha^A: \mathbb{R}^k \times \bigoplus^k E \rightarrow \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)$$

$$(5.4) \quad \mathcal{Y}_A(\mathbf{t}, \mathbf{e}_q) = (0_q; \frac{\partial}{\partial t^A}\Big|_{(\mathbf{t}, \mathbf{e}_q)}) \quad , \quad \mathcal{X}_\alpha(\mathbf{t}, \mathbf{e}_q) = (e_\alpha(q); \rho_\alpha^i(q) \frac{\partial}{\partial q^i}\Big|_{(\mathbf{t}, \mathbf{e}_q)}) \quad , \quad \mathcal{V}_\alpha^A(\mathbf{t}, \mathbf{e}_q) = (0_q; \frac{\partial}{\partial y_A^\alpha}\Big|_{(\mathbf{t}, \mathbf{e}_q)})$$

is a local basis of $\text{Sec}(\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E))$, the set of sections of $\tilde{\tau}_{\mathbb{R}^k \times \bigoplus^k E}$ (see 4.8).

(4) The anchor map $\rho^{\tilde{\rho}}: \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E) \rightarrow T(\mathbb{R}^k \times \bigoplus^k E)$ allows us to associate a vector field with each section $\xi: \mathbb{R}^k \times \bigoplus^k E \rightarrow \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)$. Locally, if

$$\xi = \xi^A \mathcal{Y}_A + \xi^\alpha \mathcal{X}_\alpha + \xi_A^\alpha \mathcal{V}_\alpha^A \in \text{Sec}(\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E))$$

then the associated vector field is given by

$$(5.5) \quad \rho^{\tilde{\rho}}(\xi) = \xi^A \frac{\partial}{\partial t^A} + \rho_\alpha^i \xi^\alpha \frac{\partial}{\partial q^i} + \xi_A^\alpha \frac{\partial}{\partial y_A^\alpha} \in \mathfrak{X}(\mathbb{R}^k \times \bigoplus^k E).$$

(5) The Lie bracket of two sections of $\tilde{\tau}_{\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)}$ is characterized by (see 4.9):

$$(5.6) \quad \begin{aligned} [\![\mathcal{Y}_A, \mathcal{Y}_B]\!]^{\tilde{\rho}} &= 0, & [\![\mathcal{Y}_A, \mathcal{X}_\alpha]\!]^{\tilde{\rho}} &= 0, & [\![\mathcal{Y}_A, \mathcal{V}_\beta^B]\!]^{\tilde{\rho}} &= 0, \\ [\![\mathcal{X}_\alpha, \mathcal{X}_\beta]\!]^{\tilde{\rho}} &= \mathcal{C}_{\alpha\beta}^\gamma \mathcal{X}_\gamma, & [\![\mathcal{X}_\alpha, \mathcal{V}_\beta^B]\!]^{\tilde{\rho}} &= 0, & [\![\mathcal{V}_\alpha^A, \mathcal{V}_\beta^B]\!]^{\tilde{\rho}} &= 0. \end{aligned}$$

(6) If $\{\mathcal{Y}^A, \mathcal{X}^\alpha, \mathcal{V}_A^\alpha\}$ is the dual basis of $\{\mathcal{Y}_A, \mathcal{X}_\alpha, \mathcal{V}_\alpha^A\}$, then the exterior differential if given locally (see 4.4) by

$$(5.7) \quad d^{\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)} f = \frac{\partial f}{\partial t^A} \mathcal{Y}^A + \rho_\alpha^i \frac{\partial f}{\partial q^i} \mathcal{X}^\alpha + \frac{\partial f}{\partial y_A^\alpha} \mathcal{V}_A^\alpha, \quad \text{for all } f \in \mathcal{C}^\infty(\mathbb{R}^k \times \bigoplus^k E)$$

$$d^{\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)} \mathcal{Y}^A = 0, \quad d^{\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)} \mathcal{X}^\alpha = -\frac{1}{2} \mathcal{C}_{\alpha\beta}^\gamma \mathcal{X}^\alpha \wedge \mathcal{X}^\beta, \quad d^{\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)} \mathcal{V}_A^\alpha = 0.$$

Remark 5.1. In the particular case $E = TQ$, the manifold $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)$ reduces to $T(\mathbb{R}^k \times T_k^1 Q)$ since

$$\begin{aligned}
 (5.8) \quad & \mathcal{T}^{TQ}(\mathbb{R}^k \times \bigoplus^k TQ) = \mathcal{T}^{TQ}(\mathbb{R}^k \times T_k^1 Q) \\
 & = \{(u_q, v_{(\mathbf{t}, \mathbf{w}_q)}) \in TQ \times T(\mathbb{R}^k \times T_k^1 Q) / u_q = Tp_Q(v_{(\mathbf{t}, \mathbf{w}_q)})\} \\
 & = \{(Tp_Q(v_{(\mathbf{t}, \mathbf{w}_q)}), v_{(\mathbf{t}, \mathbf{w}_q)}) \in TQ \times T(\mathbb{R}^k \times T_k^1 Q) / (\mathbf{t}, \mathbf{w}_q) \in \mathbb{R}^k \times T_k^1 Q\} \\
 & \equiv \{v_{(\mathbf{t}, \mathbf{w}_q)} \in T(\mathbb{R}^k \times T_k^1 Q) / (\mathbf{t}, \mathbf{w}_q) \in \mathbb{R}^k \times T_k^1 Q\} \equiv T(\mathbb{R}^k \times T_k^1 Q).
 \end{aligned}$$

◊

5.1.3. *The Liouville sections and vertical endomorphisms.* On $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)$ we define two families of canonical objects, *Liouville sections* and *vertical endomorphism* which correspond to the *Liouville vector fields* and *canonical tensor fields* on $\mathbb{R}^k \times T_k^1 Q$ (see [31, 44, 45].)

Ath-vertical lifting (see for instance [9]). An element $(a_q, v_{(\mathbf{t}, \mathbf{e}_q)})$ of $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)$ is said to be vertical if

$$\tilde{\tau}_1(a_q, v_{(\mathbf{t}, \mathbf{e}_q)}) = 0_q \in E,$$

where

$$\begin{aligned}
 \tilde{\tau}_1 : \quad & \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E) \rightarrow E, \\
 & (a_q, v_{(\mathbf{t}, \mathbf{e}_q)}) \mapsto a_q
 \end{aligned}$$

is the projection on the first factor E . The vertical elements of $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)$ are thus of the form

$$(0_q, v_{(\mathbf{t}, \mathbf{e}_q)}) \in \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)$$

where $v_{(\mathbf{t}, \mathbf{e}_q)} \in T(\mathbb{R}^k \times \bigoplus^k E)$ and $(\mathbf{t}, \mathbf{e}_q) \in \mathbb{R}^k \times \bigoplus^k E$. In particular, the tangent vector $v_{(\mathbf{t}, \mathbf{e}_q)}$ is \tilde{p} -vertical, since by (5.2)

$$0 = T_{(\mathbf{t}, \mathbf{e}_q)}\tilde{p}(v_{(\mathbf{t}, \mathbf{e}_q)}).$$

In a local coordinate system (t^A, q^i, y_A^α) on $\mathbb{R}^k \times \bigoplus^k E$, if $(a_q, v_{(\mathbf{t}, \mathbf{e}_q)}) \in \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)$ is vertical, then $a_q = 0_q$ and

$$v_{(\mathbf{t}, \mathbf{e}_q)} = u_A \frac{\partial}{\partial t^A} \Big|_{(\mathbf{t}, \mathbf{e}_q)} + u_B^\alpha \frac{\partial}{\partial y_B^\alpha} \Big|_{(\mathbf{t}, \mathbf{e}_q)} \in T_{(\mathbf{t}, \mathbf{e}_q)}(\mathbb{R}^k \times \bigoplus^k E).$$

Definition 5.2. For each $A = 1, \dots, k$, the vertical Ath-lifting is defined as the mapping

$$\begin{aligned}
 (5.9) \quad & \cdot^{\mathbf{v}_A} : E \times_Q (\mathbb{R}^k \times \bigoplus^k E) \longrightarrow \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E) \\
 & (a_q, \mathbf{t}, \mathbf{e}_q) \mapsto (a_q, \mathbf{t}, \mathbf{e}_q)^{\mathbf{v}_A} = (0_q, (a_q)_{(\mathbf{t}, \mathbf{e}_q)}^{v_A})
 \end{aligned}$$

where $a_q \in E$, $(\mathbf{t}, \mathbf{e}_q) = (t^1, \dots, t^k, e_{1q}, \dots, e_{kq}) \in \mathbb{R}^k \times \bigoplus^k E$ and the vector $(a_q)_{(\mathbf{t}, \mathbf{e}_q)}^{v_A} \in T_{(\mathbf{t}, \mathbf{e}_q)}(\mathbb{R}^k \times \bigoplus^k E)$ is given by

$$(5.10) \quad (a_q)_{(\mathbf{t}, \mathbf{e}_q)}^{v_A} f = \frac{d}{ds} \Big|_{s=0} f(\mathbf{t}, e_{1q}, \dots, e_{Aq} + sa_q, \dots, e_{kq}), \quad 1 \leq A \leq k,$$

for an arbitrary function $f \in \mathcal{C}^\infty(\mathbb{R}^k \times \bigoplus^k E)$.

The local expression of $(a_q)_{(\mathbf{t}, \mathbf{e}_q)}^{v_A}$ is

$$(5.11) \quad (a_q)_{(\mathbf{t}, \mathbf{e}_q)}^{v_A} = y^\alpha (a_q) \frac{\partial}{\partial y_A^\alpha} \Big|_{(\mathbf{t}, \mathbf{e}_q)} \in T_{(\mathbf{t}, \mathbf{e}_q)}(\mathbb{R}^k \times \bigoplus^k E), \quad 1 \leq A \leq k.$$

Since $(a_q)_{(\mathbf{t}, \mathbf{e}_q)}^{v_A} \in T_{(\mathbf{t}, \mathbf{e}_q)}(\mathbb{R}^k \times \bigoplus^k E)$ is \tilde{p} -vertical, and from (5.4), (5.9) and (5.11) we deduce that locally

$$(5.12) \quad (a_q, \mathbf{t}, \mathbf{e}_q)^{\mathbf{v}_A} = (0_q, y^\alpha (a_q) \frac{\partial}{\partial y_A^\alpha} \Big|_{(\mathbf{t}, \mathbf{e}_q)}) = y^\alpha (a_q) \mathcal{V}_\alpha^A(\mathbf{t}, \mathbf{e}_q), \quad 1 \leq A \leq k.$$

Vertical endomorphisms on $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)$. One of the most important family of canonical geometric elements on $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)$ is the family of vertical endomorphisms $\tilde{S}^1, \dots, \tilde{S}^k$. This family plays the role of the canonical tensor fields S^1, \dots, S^k in the standard case (see, for instance [31, 44, 45, 47, 48]).

Definition 5.3. For $A = 1, \dots, k$ the A th-vertical endomorphism on $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)$ is the mapping

$$(5.13) \quad \begin{aligned} \tilde{S}^A : \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E) &\rightarrow \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E) \\ (a_q, v_{(\mathbf{t}, \mathbf{e}_q)}) &\mapsto \tilde{S}^A(a_q, v_{(\mathbf{t}, \mathbf{e}_q)}) = (a_q, \mathbf{t}, \mathbf{e}_q)^{\mathbf{v}^A}, \end{aligned}$$

where $a_q \in E$, $(\mathbf{t}, \mathbf{e}_q) \in \mathbb{R}^k \times \bigoplus^k E$ and $v_{(\mathbf{t}, \mathbf{e}_q)} \in T_{(\mathbf{t}, \mathbf{e}_q)}(\mathbb{R}^k \times \bigoplus^k E)$.

Locally, let $\{\mathcal{Y}_A, \mathcal{X}_\alpha, \mathcal{V}_\alpha^A\}$ be a local basis of $\text{Sec}(\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E))$ and let $\{\mathcal{Y}^A, \mathcal{X}^\alpha, \mathcal{V}_A^\alpha\}$ be its dual basis. The corresponding local expression of \tilde{S}^A is

$$(5.14) \quad \tilde{S}^A = \sum_{\alpha=1}^m \mathcal{V}_\alpha^A \otimes \mathcal{X}^\alpha, \quad 1 \leq A \leq k.$$

Remark 5.4.

- (1) In the standard case ($E = TQ$, $\rho = id_{TQ}$), the \tilde{S}^A constitutes the canonical tensor fields S^1, \dots, S^k on $\mathbb{R}^k \times T_k^1 Q$ (see, for instance, [31, 44, 45, 47, 48]).
- (2) The endomorphisms $\tilde{S}^1, \dots, \tilde{S}^k$ defined here allows us to introduce the *Lagrangian sections* when we develop the k -cosymplectic Lagrangian formalism on Lie algebroids. Moreover these mappings give a characterization of certain sections of $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)$ which we consider in the following subsection.

◇

The Liouville sections. The A th Liouville section $\widehat{\Delta}_A$ is the section of $\tilde{\tau}_{\mathbb{R}^k \times \bigoplus^k E} : \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E) \rightarrow \mathbb{R}^k \times \bigoplus^k E$ given by

$$\begin{aligned} \widehat{\Delta}_A : \mathbb{R}^k \times \bigoplus^k E &\rightarrow \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E) \\ (\mathbf{t}, \mathbf{e}_q) &\mapsto \widehat{\Delta}_A(\mathbf{t}, \mathbf{e}_q) = (pr_A(\mathbf{t}, \mathbf{e}_q), \mathbf{t}, \mathbf{e}_q)^{\mathbf{v}^A} = (e_{Aq}, \mathbf{t}, \mathbf{e}_q)^{\mathbf{v}^A}, \end{aligned}$$

where $pr_A : \mathbb{R}^k \times \bigoplus^k E \rightarrow E$ is the canonical projection of $\mathbb{R}^k \times \bigoplus^k E$ over the A th copy of E . From the local expression (5.12) of \mathbf{v}^A , and since

$$y^\alpha(e_{Aq}) = y_A^\alpha(\mathbf{t}, e_{1q}, \dots, e_{kq}) = y_A^\alpha(\mathbf{t}, \mathbf{e}_q),$$

$\widehat{\Delta}_A$ has the local expression

$$(5.15) \quad \widehat{\Delta}_A = \sum_{\alpha=1}^m y_A^\alpha \mathcal{V}_\alpha^A, \quad 1 \leq A \leq k.$$

Remark 5.5. In the standard case, $\widehat{\Delta}_A$ is the A th-Liouville vector field Δ_A on $\mathbb{R}^k \times T_k^1 Q$, (see for instance [31, 44, 47, 48]). ◇

In the standard Lagrangian k -cosymplectic formalism, the Liouville vector fields $\Delta_1, \dots, \Delta_k$ allows us to define the energy function. Analogously as we will see below, the energy function can be defined in the Lie algebroid setting using the Liouville sections $\widehat{\Delta}_1, \dots, \widehat{\Delta}_k$.

5.1.4. Second order partial differential equations (SOPDE's). In the standard k -cosymplectic Lagrangian formalism one obtains the solutions of the Euler-Lagrange equations as integral sections of certain second-order partial differential equations (SOPDE in the sequel) on $\mathbb{R}^k \times T_k^1 Q$. In order to introduce the analogous object on Lie algebroids, we note that in the standard case a SOPDE ξ is a k -vector field on $\mathbb{R}^k \times T_k^1 Q$, that is, a section of

$$T_k^1(\mathbb{R}^k \times T_k^1 Q) \equiv T(\mathbb{R}^k \times T_k^1 Q) \oplus \dots \oplus T(\mathbb{R}^k \times T_k^1 Q) \rightarrow \mathbb{R}^k \times T_k^1 Q,$$

which satisfies certain properties. Since $T_k^1(\mathbb{R}^k \times T_k^1 Q)$ is the Whitney sum of k copies of $T(\mathbb{R}^k \times T_k^1 Q)$, it is natural to think that, in the Lie algebroid context, the appropriate space would be the Whitney sum of k copies of $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E)$, that is

$$(\mathcal{T}^E)_k^1(\mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E) = \mathcal{T}^E(\mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E) \oplus \dots \oplus \mathcal{T}^E(\mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E).$$

We denote by $\tilde{\tau}_{\mathbb{R}^k \times \oplus E}^k$ its canonical projection on $\mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E$.

Definition 5.6. A second order partial differential equation (SOPDE) on $\mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E$ is a map $\xi = (\xi_1, \dots, \xi_k) : \mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E \rightarrow (\mathcal{T}^E)_k^1(\mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E)$ which is a section of $\tilde{\tau}_{\mathbb{R}^k \times \oplus E}^k$ and satisfies the equations

$$\tilde{S}^A(\xi_A) = \hat{\Delta}_A \quad \text{and} \quad \mathcal{Y}^B(\xi_A) = \delta_A^B, \quad 1 \leq A, B \leq k.$$

Since $(\mathcal{T}^E)_k^1(\mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E)$ is the Whitney sum of k copies of $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E)$, we deduce that to give a section ξ of $\tilde{\tau}_{\mathbb{R}^k \times \oplus E}^k$ is equivalent to giving a family of k sections ξ_1, \dots, ξ_k of the Lagrangian prolongation $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E)$ obtained by projection ξ on each factor.

One easily deduces that the local expression of a SOPDE $\xi = (\xi_1, \dots, \xi_k)$ is

$$(5.16) \quad \xi_A = \mathcal{Y}_A + y_A^\alpha \mathcal{X}_\alpha + (\xi_A)_B^\alpha \mathcal{V}_\alpha^B,$$

where $(\xi_A)_\alpha^B \in \mathcal{C}^\infty(\mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E)$.

Lemma 5.7. Let $\xi = (\xi_1, \dots, \xi_k) : \mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E \rightarrow (\mathcal{T}^E)_k^1(\mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E)$ be a section of $\tilde{\tau}_{\mathbb{R}^k \times \oplus E}^k$. Then

$$(\rho^{\tilde{P}}(\xi_1), \dots, \rho^{\tilde{P}}(\xi_k)) : \mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E \rightarrow T_k^1(\mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E)$$

is a k -vector field on $\mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E$, where $\rho^{\tilde{P}} : \mathcal{T}^E(\mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E) \equiv E \times_{TQ} T(\mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E) \rightarrow T(\mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E)$ is the anchor map of the Lie algebroid $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E)$.

Proof. Directly by section 5.1.2 (6). ■

In local coordinates

$$(5.17) \quad \rho^{\tilde{P}}(\xi_A) = \frac{\partial}{\partial t^A} + \rho_\alpha^i y_A^\alpha \frac{\partial}{\partial q^i} + (\xi_A)_B^\alpha \frac{\partial}{\partial y_B^\alpha} \in \mathfrak{X}(\mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E).$$

Definition 5.8. A map $\eta : U \subseteq \mathbb{R}^k \rightarrow \mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E$ is an integral section of a SOPDE $\xi = (\xi_1, \dots, \xi_k)$ if η is an integral section of the k -vector field $(\rho^{\tilde{P}}(\xi_1), \dots, \rho^{\tilde{P}}(\xi_k))$ associated with ξ , that is,

$$(5.18) \quad \rho^{\tilde{P}}(\xi_A)(\eta(\mathbf{t})) = \eta_*(\mathbf{t}) \left(\frac{\partial}{\partial t^A} \Big|_{\mathbf{t}} \right), \quad 1 \leq A \leq k.$$

In η is written locally as $\eta(\mathbf{t}) = (\eta_A(\mathbf{t}), \eta^i(\mathbf{t}), \eta_A^\alpha(\mathbf{t}))$, then from (5.17) we deduce that (5.18) is locally equivalent to the identities

$$(5.19) \quad \frac{\partial \eta_B}{\partial t^A} \Big|_{\mathbf{t}} = \delta_B^A, \quad \frac{\partial \eta^i}{\partial t^A} \Big|_{\mathbf{t}} = \eta_A^\alpha(\mathbf{t}) \rho_\alpha^i, \quad \frac{\partial \eta_B^\alpha}{\partial t^A} \Big|_{\mathbf{t}} = (\xi_A)_B^\alpha(\eta(\mathbf{t})).$$

5.1.5. Lagrangian formalism. In this section we develop an intrinsic and global geometric framework that allows us to write the Euler-Lagrange equations associated with a Lagrangian function $L : \mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E \rightarrow \mathbb{R}$ on a Lie algebroid. We first introduce some geometric elements associated with L .

Poincaré-Cartan or Lagrangian sections. The Poincaré-Cartan 1-sections Θ_L^A are defined by

$$\begin{aligned} \Theta_L^A : \mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E &\longrightarrow (\mathcal{T}^E(\mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E))^* \\ (\mathbf{t}, \mathbf{e}_q) &\longmapsto \Theta_L^A(\mathbf{t}, \mathbf{e}_q) \end{aligned}$$

where $\Theta_L^A(\mathbf{t}, \mathbf{e}_q) : (\mathcal{T}^E(\mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E))_{(\mathbf{t}, \mathbf{e}_q)} \rightarrow \mathbb{R}$ is the linear mapping defined by

$$(5.20) \quad (\Theta_L^A)(\mathbf{t}, \mathbf{e}_q)(a_q, v_{(\mathbf{t}, \mathbf{e}_q)}) = (d^{\mathcal{T}^E(\mathbb{R}^k \times \bigoplus_{\mathbb{R}^k \times \oplus E}^k E)} L)_{(\mathbf{t}, \mathbf{e}_q)}((\tilde{S}^A)_{(\mathbf{t}, \mathbf{e}_q)}(a_q, v_{(\mathbf{t}, \mathbf{e}_q)})).$$

Using (5.7) with $f = L$,

$$(5.21) \quad \begin{aligned} (\Theta_L^A)(\mathbf{t}, \mathbf{e}_q)(a_q, v_{(\mathbf{t}, \mathbf{e}_q)}) &= (d^{\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)} L)_{(\mathbf{t}, \mathbf{e}_q)}((\tilde{S}^A)_{(\mathbf{t}, \mathbf{e}_q)}(a_q, v_{(\mathbf{t}, \mathbf{e}_q)})) \\ &= [\rho^{\tilde{P}}((\tilde{S}^A)_{(\mathbf{t}, \mathbf{e}_q)}(a_q, v_{(\mathbf{t}, \mathbf{e}_q)}))]L, \end{aligned}$$

where $(\mathbf{t}, \mathbf{e}_q) \in \mathbb{R}^k \times \bigoplus^k E$, $(a_q, v_{(\mathbf{t}, \mathbf{e}_q)}) \in [\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)]_{(\mathbf{t}, \mathbf{e}_q)}$ and

$$\rho^{\tilde{P}}((\tilde{S}^A)_{(\mathbf{t}, \mathbf{e}_q)}(a_q, v_{(\mathbf{t}, \mathbf{e}_q)})) \in T_{(\mathbf{t}, \mathbf{e}_q)}(\mathbb{R}^k \times \bigoplus^k E).$$

The Poincaré-Cartan 2-sections

$$\Omega_L^A : \mathbb{R}^k \times \bigoplus^k E \rightarrow \Lambda^2(\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E))^*, \quad 1 \leq A \leq k$$

are defined by

$$\Omega_L^A := -d^{\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)} \Theta_L^A, \quad 1 \leq A \leq k.$$

To find the local expression of Θ_L^A and Ω_L^A , consider $\{\mathcal{Y}_B, \mathcal{X}_\alpha, \mathcal{V}_\alpha^B\}$, a local basis of $\text{Sec}(\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E))$, and its dual basis $\{\mathcal{Y}^B, \mathcal{X}^\alpha, \mathcal{V}_B^\alpha\}$. From (5.5), (5.14) and (5.21), we deduce that

$$(5.22) \quad \Theta_L^A = \frac{\partial L}{\partial y_A^\alpha} \mathcal{X}^\alpha, \quad 1 \leq A \leq k,$$

and from the local expressions (5.5), (5.6), (5.7) and (5.22),

$$(5.23) \quad \Omega_L^A = \frac{1}{2} \left(\rho_\beta^i \frac{\partial^2 L}{\partial q^i \partial y_A^\alpha} - \rho_\alpha^i \frac{\partial^2 L}{\partial q^i \partial y_A^\beta} + \mathcal{C}_{\alpha\beta}^\gamma \frac{\partial L}{\partial y_A^\gamma} \right) \mathcal{X}^\alpha \wedge \mathcal{X}^\beta + \frac{\partial^2 L}{\partial t^B \partial y_A^\alpha} \mathcal{X}^\alpha \wedge \mathcal{Y}^B + \frac{\partial^2 L}{\partial y_B^\beta \partial y_A^\alpha} \mathcal{X}^\alpha \wedge \mathcal{V}_B^\beta.$$

We say that the lagrangian L is *regular* if the matrix $(\frac{\partial^2 L}{\partial y_A^\alpha \partial y_B^\beta})$ is non-singular.

Remark 5.9. When we consider the particular case $E = TQ$ and $\rho = id_{TQ}$,

$$\Omega_L^A(X, Y) = \omega_L^A(X, Y), \quad 1 \leq A \leq k,$$

where X, Y are vector fields on $\mathbb{R}^k \times T_k^1 Q$ and $\omega_L^1, \dots, \omega_L^k$ are the Lagrangian 2-forms of the standard k -cosymplectic Lagrangian formalism, see for instance [44, 47, 48]. \diamond

The energy function. The *energy function* $E_L : \mathbb{R}^k \times \bigoplus^k E \rightarrow \mathbb{R}$ defined by the Lagrangian L is

$$E_L = \sum_{A=1}^k \rho^{\tilde{P}}(\hat{\Delta}_A)L - L.$$

and from (5.5) and (5.15) one deduces that E_L is locally given by

$$(5.24) \quad E_L = \sum_{A=1}^k y_A^\alpha \frac{\partial L}{\partial y_A^\alpha} - L \in \mathcal{C}^\infty(\mathbb{R}^k \times \bigoplus^k E).$$

Morphisms. We generalize the Euler-Lagrange equations and their solutions to the case of Lie algebroids in terms of Lie algebroid morphisms.

In the standard Lagrangian k -cosymplectic formalism, a solution of the Euler-Lagrange equations is a field $\phi : \mathbb{R}^k \rightarrow Q$ with a first prolongation $\phi^{[1]} : \mathbb{R}^k \rightarrow \mathbb{R}^k \times T_k^1 Q$ satisfying those equations, that is,

$$\sum_{A=1}^k \frac{\partial}{\partial t^A} \Big|_{\mathbf{t}} \left(\frac{\partial L}{\partial v_A^i} \Big|_{\phi^{[1]}(\mathbf{t})} \right) = \frac{\partial L}{\partial q^i} \Big|_{\phi^{[1]}(\mathbf{t})}.$$

The map ϕ naturally induces the Lie algebroid morphism

$$\begin{array}{ccc} T\mathbb{R}^k & \xrightarrow{T\phi} & TQ \\ \tau_{\mathbb{R}^k} \downarrow & & \downarrow \tau_Q \\ \mathbb{R}^k & \xrightarrow{\phi} & Q \end{array}$$

and in terms of the canonical basis of sections of $\tau_{\mathbb{R}^k}$, $\left\{ \frac{\partial}{\partial t^1}, \dots, \frac{\partial}{\partial t^k} \right\}$, the first prolongation of ϕ , $\phi^{[1]}$, can be written as

$$\phi^{[1]}(\mathbf{t}) = (\mathbf{t}, T_{\mathbf{t}}\phi(\frac{\partial}{\partial t^1}|_{\mathbf{t}}), \dots, T_{\mathbf{t}}\phi(\frac{\partial}{\partial t^k}|_{\mathbf{t}})).$$

For a general Lie algebroid we shall derive Euler-Lagrange equations for field theories on Lie algebroids using as a main tool Lie algebroid morphisms $\Phi = (\bar{\Phi}, \underline{\Phi})$,

$$\begin{array}{ccc} T\mathbb{R}^k & \xrightarrow{\bar{\Phi}} & E \\ \tau_{\mathbb{R}^k} \downarrow & & \downarrow \tau \\ \mathbb{R}^k & \xrightarrow{\underline{\Phi}} & Q \end{array}$$

with an associated map $\tilde{\Phi} : \mathbb{R}^k \rightarrow \mathbb{R}^k \times \bigoplus^k E$

$$\begin{aligned} \tilde{\Phi} : \mathbb{R}^k &\rightarrow \mathbb{R}^k \times \bigoplus^k E \equiv \mathbb{R}^k \times E \oplus \dots \oplus E \\ \mathbf{t} &\rightarrow (\mathbf{t}, \bar{\Phi}(e_1(\mathbf{t})), \dots, \bar{\Phi}(e_k(\mathbf{t}))) \end{aligned}$$

where $\{e_A\}_{A=1}^k$ is a fixed local basis of local sections of $T\mathbb{R}^k$.

If (t^A) and (q^i) are local coordinate systems on \mathbb{R}^k and Q , respectively; $\{e_A\}$ and $\{e_\alpha\}$ local basis of sections of $\tau_{\mathbb{R}^k}$ and E , respectively; and $\{e^A\}$ and $\{e^\alpha\}$ the respective dual bases; then $\underline{\Phi}(\mathbf{t}) = (\phi^i(\mathbf{t}))$ and $\Phi^*e^\alpha = \phi_A^\alpha e^A$ for certain local functions ϕ^i and ϕ_A^α on \mathbb{R}^k , the associated map $\tilde{\Phi}$ is given locally by $\tilde{\Phi}(\mathbf{t}) = (t^A, \phi^i(\mathbf{t}), \phi_A^\alpha(\mathbf{t}))$, and the Lie algebroid morphism conditions (4.7) are

$$(5.25) \quad \rho_\alpha^i \phi_A^\alpha = \frac{\partial \phi^i}{\partial t^A} \quad , \quad 0 = \frac{\partial \phi_A^\alpha}{\partial t^B} - \frac{\partial \phi_B^\alpha}{\partial t^A} + C_{\beta\gamma}^\alpha \phi_B^\beta \phi_A^\gamma .$$

Remark 5.10. In the standard case ($E = TQ$), the morphism conditions reduce to

$$\phi_A^i = \frac{\partial \phi^i}{\partial t^A} \quad \text{and} \quad \frac{\partial \phi_A^i}{\partial t^B} = \frac{\partial \phi_B^i}{\partial t^A} ,$$

i.e., the standard first-order prolongations of fields $\phi : \mathbb{R}^k \rightarrow Q$. \diamond

The Euler-Lagrange equations. Given a regular Lagrangian function $L : \mathbb{R}^k \times \bigoplus^k E \rightarrow \mathbb{R}$, it is natural to consider sections $\xi_L = (\xi_1, \dots, \xi_k)$ of $(\mathcal{T}^E)_k^1(\mathbb{R}^k \times \bigoplus^k E) = \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E) \oplus \dots \oplus \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E) \rightarrow \mathbb{R}^k \times \bigoplus^k E$ such that

$$(5.26) \quad \mathcal{Y}^B(\xi_A) = \delta_A^B \quad , \quad \sum_{A=1}^k \iota_{\xi_A} \Omega_L^A = d^{\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)} E_L + \sum_{A=1}^k \frac{\partial L}{\partial t^A} \mathcal{Y}^A .$$

equation (5.26) being the analog of the geometric Euler-Lagrange equations of the standard k -cosymplectic Lagrangian formalism.

Theorem 5.11. *Let $L : \mathbb{R}^k \times \bigoplus^k E \rightarrow \mathbb{R}$ be a regular Lagrangian, and ξ_1, \dots, ξ_k k sections of $\tilde{\tau}_{\mathbb{R}^k \times \bigoplus^k E} : \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E) \rightarrow \mathbb{R}^k \times \bigoplus^k E$ such that*

$$\mathcal{Y}^B(\xi_A) = \delta_A^B \quad , \quad \sum_{A=1}^k \iota_{\xi_A} \Omega_L^A = d^{\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)} E_L + \sum_{A=1}^k \frac{\partial L}{\partial t^A} \mathcal{Y}^A .$$

Then:

- (1) $\xi_L = (\xi_1, \dots, \xi_k)$ is a SOPDE.
- (2) If $\tilde{\Phi} : \mathbb{R}^k \rightarrow \mathbb{R}^k \times \bigoplus^k E$ is the map associated with a Lie algebroid morphism between $T\mathbb{R}^k$ and E , and is an integral section of ξ_L , then it is a solution of the Euler-Lagrange equations of field

theories on Lie algebroids, that is,

$$\begin{aligned}
 (5.27) \quad \sum_{A=1}^k \frac{\partial}{\partial t^A} \left(\frac{\partial L}{\partial y_A^\alpha} \Big|_{\tilde{\Phi}(\mathbf{t})} \right) &= \rho_\alpha^i \frac{\partial L}{\partial q^i} \Big|_{\tilde{\Phi}(\mathbf{t})} - \phi_A^\beta(\mathbf{t}) \mathcal{C}_{\alpha\beta}^\gamma \frac{\partial L}{\partial y_A^\gamma} \Big|_{\tilde{\Phi}(\mathbf{t})}, \\
 \frac{\partial \phi^i}{\partial t^A} \Big|_{\mathbf{t}} &= \phi_A^\alpha(\mathbf{t}) \rho_\alpha^i \\
 0 &= \frac{\partial \phi_A^\alpha}{\partial t^B} \Big|_{\mathbf{t}} - \frac{\partial \phi_B^\alpha}{\partial t^A} \Big|_{\mathbf{t}} + \mathcal{C}_{\beta\gamma}^\alpha \phi_B^\beta(\mathbf{t}) \phi_A^\gamma(\mathbf{t}).
 \end{aligned}$$

Proof. The proof is analogous to the one in Theorem 4.18 in [24].

In this case one obtains that if $\xi_L = (\xi_1, \dots, \xi_k) : \mathbb{R}^k \times \bigoplus^k E \rightarrow (\mathcal{T}^E)_k^1(\mathbb{R}^k \times \bigoplus^k E)$ is a solution to (5.26) then:

(1) ξ_L is a SOPDE on $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E)$. With respect to a local coordinate system (t^A, q^i, y_A^α) on $\mathbb{R}^k \times \bigoplus^k E$ and a local basis $\{e_\alpha\}$ of $\text{Sec}(E)$ it is given locally by

$$\xi_A = \mathcal{Y}_A + y_A^\alpha \mathcal{X}_\alpha + (\xi_A)_B^\alpha \mathcal{V}_\alpha^B$$

$(\xi_A)_B^\alpha$ being functions on $\mathbb{R}^k \times \bigoplus^k E$;

(2) the functions $(\xi_A)_B^\alpha \in \mathcal{C}^\infty(\mathbb{R}^k \times \bigoplus^k E)$ satisfy the following equations:

$$(5.28) \quad \frac{\partial^2 L}{\partial t^A \partial y_A^\alpha} + y_A^\beta \rho_\beta^i \frac{\partial^2 L}{\partial q^i \partial y_A^\alpha} + (\xi_A)_B^\beta \frac{\partial^2 L}{\partial y_B^\beta \partial y_A^\alpha} = \rho_\alpha^i \frac{\partial L}{\partial q^i} - y_A^\beta \mathcal{C}_{\alpha\beta}^\gamma \frac{\partial L}{\partial y_A^\gamma}.$$

If the map $\tilde{\Phi} : \mathbb{R}^k \rightarrow \mathbb{R}^k \times \bigoplus^k E$ associated with a Lie algebroid morphism $\Phi : T\mathbb{R}^k \rightarrow E$ and defined by $\tilde{\Phi}(\mathbf{t}) = (\mathbf{t}, \phi^i(\mathbf{t}), \phi_A^\alpha(\mathbf{t}))$, is an integral section of ξ_L , then by condition (5.19) and equations (5.28) we obtain

$$\begin{aligned}
 \sum_{A=1}^k \frac{\partial}{\partial t^A} \left(\frac{\partial L}{\partial y_A^\alpha} \Big|_{\tilde{\Phi}(\mathbf{t})} \right) &= \rho_\alpha^i \frac{\partial L}{\partial q^i} \Big|_{\tilde{\Phi}(\mathbf{t})} - \phi_A^\beta(\mathbf{t}) \mathcal{C}_{\alpha\beta}^\gamma \frac{\partial L}{\partial y_A^\gamma} \Big|_{\tilde{\Phi}(\mathbf{t})}, \\
 \frac{\partial \phi^i}{\partial t^A} \Big|_{\mathbf{t}} &= \phi_A^\alpha(\mathbf{t}) \rho_\alpha^i, \\
 0 &= \frac{\partial \phi_A^\alpha}{\partial t^B} \Big|_{\mathbf{t}} - \frac{\partial \phi_B^\alpha}{\partial t^A} \Big|_{\mathbf{t}} + \mathcal{C}_{\beta\gamma}^\alpha \phi_B^\beta(\mathbf{t}) \phi_A^\gamma(\mathbf{t})
 \end{aligned}$$

where the last two equations are consequence of the morphism conditions (5.25). \blacksquare

If E is the standard Lie algebroid TQ , the previous equations are the classical Euler-Lagrange equations for the Lagrangian $L : \mathbb{R}^k \times T_k^1 Q \rightarrow \mathbb{R}$. In what follows (5.27) will be called the *Euler-Lagrange equations of field theories on Lie algebroids*.

Remark 5.12.

- (1) Equations (5.27) are obtained by E. Martínez [40] using a variational approach in the multisymplectic framework.
- (2) If L does not depends on \mathbf{t} , then it can be considered as a map $L : \bigoplus^k E \rightarrow \mathbb{R}$. In this case the sections Ω_L^A can be thought as sections of $\mathcal{T}^E(\bigoplus^k E)$ and from (5.26) we deduce the k -symplectic Euler-Lagrange equations on Lie algebroids developed in [24].
- (3) When $E = TQ$, equations (5.26) are the standard k -cosymplectic geometric version of the Euler-Lagrange equations for field theories develop by M. de León *et al* in [31].
- (4) When L does not depends on \mathbf{t} and $E = TQ$ and $\rho = id_{TQ}$, equations (5.26) coincide with the Euler-Lagrange equations of the Günther formalism [19].

\diamond

In the following table we write the geometric Lagrangian equations in the above particular cases.

LAGRANGIAN FORMALISM Geometric Lagrangian equations	
k -cosymplectic formalism on Lie algebroids	$\mathcal{Y}^B(\xi_A) = \delta_A^B$ $\sum_{A=1}^k \iota_{\xi_A} \Omega_L^A = d^{\mathcal{T}^E(\mathbb{R}^k \times \bigoplus E)} E_L + \sum_{A=1}^k \frac{\partial L}{\partial t^A} \mathcal{Y}_A$ $(\xi_1, \dots, \xi_k) \text{ family of } k \text{ sections of } \mathcal{T}^E(\mathbb{R}^k \times \bigoplus E)$
k -symplectic formalism on Lie algebroids $(\frac{\partial L}{\partial t^A} = 0, A = 1, \dots, k)$	$\sum_{A=1}^k \iota_{\xi_A} \Omega_L^A = d^{\mathcal{T}^E(\bigoplus E)} E_L$ $(\xi_1, \dots, \xi_k) \text{ family of } k \text{ sections of } \mathcal{T}^E(\bigoplus E)$
Standard k -cosymplectic formalism $(E = TQ)$	$dt^A(Y_B) = \delta_B^A$ $\sum_{A=1}^k i_{Y_A} \omega_L^A = dE_L + \sum_{A=1}^k \frac{\partial L}{\partial t^A} dt^A$ $(Y_1, \dots, Y_k) \text{ k-vector field on } \mathbb{R}^k \times T_k^1 Q$
Standard k -symplectic formalism $(E = TQ)$ $(\frac{\partial L}{\partial t^A} = 0, A = 1, \dots, k)$	$\sum_{A=1}^k i_{Y_A} \omega_L^A = dE_L$ $(Y_1, \dots, Y_k) \text{ k-vector field on } T_k^1 Q$

Remark 5.13. When $k = 1$,

- (1) If L explicitly depends on t , equations (5.26) are the equations of Lagrangian mechanics for time-dependent system defined on Lie algebroids, see for instance [51, 52]. Moreover. In this case, when $E = TQ$, (5.26) are the dynamical equations of non-autonomous mechanics (see [11]).
- (2) If L does not depends on t , equations (5.26) are the geometric equations for autonomous lagrangian mechanics on Lie algebroids, see for instance [38]. Finally in this case if $E = TQ$ we have the classical equations for autonomous mechanics.

◊

5.2. Hamiltonian formalism. In this section we extend the standard Hamiltonian k -cosymplectic formalism to Lie algebroids. In the following, we consider a Lie algebroid $(E, [\cdot, \cdot]_E, \rho_E)$ over a manifold Q , and the dual bundle, $\tau^* : E^* \rightarrow Q$ of E .

5.2.1. *The manifold $\mathbb{R}^k \times \bigoplus^k E^*$.* The appropriate space of the standard Hamiltonian k -cosymplectic formalism is the bundle $\mathbb{R}^k \times (T_k^1)^* Q$, where $(T_k^1)^* Q$ is the bundle of k^1 -velocities of Q , that is, the Whitney sum of k copies of $T^* Q$. For this generalization to Lie algebroids, it is natural to consider that the analog of $\mathbb{R}^k \times (T_k^1)^* Q$ is

$$\mathbb{R}^k \times \bigoplus^k E^* \equiv \mathbb{R}^k \times (E^* \oplus \dots \oplus E^*),$$

with the projection map

$$\tilde{p}^* : \mathbb{R}^k \times \bigoplus^k E^* \rightarrow Q, \quad \tilde{p}^*(t^1, \dots, t^k, e_{1q}^*, \dots, e_{kq}^*) = q,$$

$\bigoplus^k E^*$ being the Whitney sum of k copies of the dual space E^* .

Let us observe that the elements of $\mathbb{R}^k \times \bigoplus^k E^*$ are of the form

$$(\mathbf{t}, \mathbf{e}_q^*) = (t^1, \dots, t^k, e_{1q}^*, \dots, e_{kq}^*).$$

If (q^i, y_α) are local coordinates on $(\tau^*)^{-1}(U) \subseteq E^*$, then the induced local coordinates (t^A, q^i, y_α^A) on $(\tilde{p}^*)^{-1}(U) \subseteq \mathbb{R}^k \times \bigoplus^k E^*$ are given by

$$(5.29) \quad t^A(\mathbf{t}, \mathbf{e}_q^*) = t^A(\mathbf{t}), \quad q^i(\mathbf{t}, \mathbf{e}_q^*) = q^i(q), \quad y_\alpha^A(\mathbf{t}, \mathbf{e}_q^*) = y_\alpha(e_{Aq}^*).$$

5.2.2. *The Hamiltonian prolongation.* We next consider the prolongation of a Lie algebroid E over the fibration $\tilde{p}^* : \mathbb{R}^k \times \bigoplus^k E^* \rightarrow Q$, that is (see section 4.4)

$$(5.30) \quad \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*) = \{(a_q, v_{(\mathbf{t}, \mathbf{e}_q^*)}) \in E \times T(\mathbb{R}^k \times \bigoplus^k E^*) / \rho(a_q) = T\tilde{p}^*(v_{(\mathbf{t}, \mathbf{e}_q^*)})\}.$$

Taking into account the description of the prolongation $\mathcal{T}^E P$ and the results on Section 4.4 (see also [9, 25, 37]), we obtain

(1) $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*) \subset E \times T(\mathbb{R}^k \times \bigoplus^k E^*)$ is a Lie algebroid over $\mathbb{R}^k \times \bigoplus^k E^*$, with the projection

$$\tilde{\tau}_{\mathbb{R}^k \times \bigoplus^k E^*} : \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*) \longrightarrow \mathbb{R}^k \times \bigoplus^k E^*$$

and Lie algebroid structure $([\![\cdot, \cdot]\!]^{\tilde{p}^*}, \rho^{\tilde{p}^*})$, where the anchor map

$$\rho^{\tilde{p}^*} : \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*) \rightarrow T(\mathbb{R}^k \times \bigoplus^k E^*)$$

is the canonical projection onto the second factor. We refer to this Lie algebroid as the k -*cosymplectic Hamiltonian prolongation*

(2) Local coordinates (t^A, q^i, y_α^A) on $\mathbb{R}^k \times \bigoplus^k E^*$ induce local coordinates $(t^A, q^i, y_\alpha^A, z^\alpha, v_A, w_\alpha^A)$ on $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)$, where

$$(5.31) \quad \begin{aligned} t^A(a_q, v_{(\mathbf{t}, \mathbf{e}_q^*)}) &= t^A(\mathbf{t}), & z^\alpha(a_q, v_{(\mathbf{t}, \mathbf{e}_q^*)}) &= y^\alpha(a_q), \\ q^i(a_q, v_{(\mathbf{t}, \mathbf{e}_q^*)}) &= q^i(q), & v_A(a_q, v_{(\mathbf{t}, \mathbf{e}_q^*)}) &= v_{(\mathbf{t}, \mathbf{e}_q^*)}(t^A), \\ y_\alpha^A(a_q, v_{(\mathbf{t}, \mathbf{e}_q^*)}) &= y_\alpha^A(\mathbf{t}, \mathbf{e}_q^*), & w_\alpha^A(a_q, v_{(\mathbf{t}, \mathbf{e}_q^*)}) &= v_{(\mathbf{t}, \mathbf{e}_q^*)}(y_\alpha^A). \end{aligned}$$

(3) The set $\{\mathcal{Y}_A, \mathcal{X}_\alpha, \mathcal{V}_A^\alpha\}$ given by

$$\mathcal{Y}_A, \mathcal{X}_\alpha, \mathcal{V}_A^\alpha : \mathbb{R}^k \times \bigoplus^k E^* \rightarrow \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)$$

$$(5.32) \quad \mathcal{Y}_A(\mathbf{t}, \mathbf{e}_q^*) = (0_q; \frac{\partial}{\partial t^A} \Big|_{(\mathbf{t}, \mathbf{e}_q^*)}), \quad \mathcal{X}_\alpha(\mathbf{t}, \mathbf{e}_q^*) = (e_\alpha(q); \rho_\alpha^i(q) \frac{\partial}{\partial q^i} \Big|_{(\mathbf{t}, \mathbf{e}_q^*)}), \quad \mathcal{V}_A^\alpha(\mathbf{t}, \mathbf{e}_q^*) = (0_q; \frac{\partial}{\partial y_\alpha^A} \Big|_{(\mathbf{t}, \mathbf{e}_q^*)})$$

is a local basis of $\text{Sec}(\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*))$, the set of sections of $\tilde{\tau}_{\mathbb{R}^k \times \bigoplus^k E^*}$ (see (4.8)).

(4) The anchor map $\rho^{\tilde{p}^*} : \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*) \rightarrow T(\mathbb{R}^k \times \bigoplus^k E^*)$ allows us to associate a vector field with each section $\xi : \mathbb{R}^k \times \bigoplus^k E^* \rightarrow \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)$ of $\tilde{\tau}_{\mathbb{R}^k \times \bigoplus^k E^*}$. Locally, if ξ is given by

$$\xi = \xi^A \mathcal{Y}_A + \xi^\alpha \mathcal{X}_\alpha + \xi_\alpha^A \mathcal{V}_A^\alpha \in \text{Sec}(\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)),$$

then the associate vector field is

$$(5.33) \quad \rho^{\tilde{p}^*}(\xi) = \xi^A \frac{\partial}{\partial t^A} + \rho_\alpha^i \xi^\alpha \frac{\partial}{\partial q^i} + \xi_\alpha^A \frac{\partial}{\partial y_\alpha^A} \in \mathfrak{X}(\mathbb{R}^k \times \bigoplus^k E^*).$$

(5) The Lie bracket of two sections of $\tilde{\tau}_{\mathbb{R}^k \times \bigoplus^k E^*}$ is characterized by the relations (see (4.9)),

$$(5.34) \quad \begin{aligned} [\![\mathcal{Y}_A, \mathcal{Y}_B]\!]^{\tilde{p}^*} &= 0, & [\![\mathcal{Y}_A, \mathcal{X}_\alpha]\!]^{\tilde{p}^*} &= 0, & [\![\mathcal{Y}_A, \mathcal{V}_B^\beta]\!]^{\tilde{p}^*} &= 0, \\ [\![\mathcal{X}_\alpha, \mathcal{X}_\beta]\!]^{\tilde{p}^*} &= \mathcal{C}_{\alpha\beta}^\gamma \mathcal{X}_\gamma, & [\![\mathcal{X}_\alpha, \mathcal{V}_B^\beta]\!]^{\tilde{p}^*} &= 0, & [\![\mathcal{V}_A^\alpha, \mathcal{V}_B^\beta]\!]^{\tilde{p}^*} &= 0. \end{aligned}$$

(6) If $\{\mathcal{Y}^A, \mathcal{X}^\alpha, \mathcal{V}_\alpha^A\}$ is the dual basis of $\{\mathcal{Y}_A, \mathcal{X}_\alpha, \mathcal{V}_A^\alpha\}$, then the exterior differential is given by

$$(5.35) \quad \begin{aligned} d^{\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)} f &= \frac{\partial f}{\partial t^A} \mathcal{Y}^A + \rho_\alpha^i \frac{\partial f}{\partial q^i} \mathcal{X}^\alpha + \frac{\partial f}{\partial y_\alpha^A} \mathcal{V}_\alpha^A, \quad \text{for all } f \in \mathcal{C}^\infty(\mathbb{R}^k \times \bigoplus^k E^*) \\ d^{\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)} \mathcal{Y}^A &= 0, \quad d^{\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)} \mathcal{X}^\gamma = -\frac{1}{2} \mathcal{C}_{\alpha\beta}^\gamma \mathcal{X}^\alpha \wedge \mathcal{X}^\beta, \quad d^{\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)} \mathcal{V}_\gamma^A = 0, \end{aligned}$$

(see (4.10)).

Remark 5.14. In the particular case $E = TQ$, the manifold $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)$ reduces to $T(\mathbb{R}^k \times (T_k^1)^*Q)$. The proof is analogous to the one in remark 5.1. \diamond

5.2.3. *The vector bundle $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*) \oplus \dots \oplus \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)$.* In the standard Hamiltonian k -cosymplectic formalism one obtains the solutions of the Hamilton equations as integral sections of certain k -vector fields on $\mathbb{R}^k \times (T_k^1)^*Q$, that is sections of

$$\tau_{\mathbb{R}^k \times (T_k^1)^*Q}^k : T_k^1(\mathbb{R}^k \times (T_k^1)^*Q) \rightarrow \mathbb{R}^k \times (T_k^1)^*Q.$$

Since on Lie algebroids the vector bundle $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)$ plays the role of $T(\mathbb{R}^k \times (T_k^1)^*Q)$, it is natural to assume that the role of

$$T_k^1(\mathbb{R}^k \times (T_k^1)^*Q) \equiv T(\mathbb{R}^k \times (T_k^1)^*Q) \oplus \dots \oplus T(\mathbb{R}^k \times (T_k^1)^*Q)$$

is played by

$$(\mathcal{T}^E)_k^1(\mathbb{R}^k \times \bigoplus^k E^*) : = \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*) \oplus \dots \oplus \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*),$$

the Whitney sum of k copies of $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)$, being the canonical projection $\tilde{\tau}_{\mathbb{R}^k \times \bigoplus^k E^*}^k : (\mathcal{T}^E)_k^1(\mathbb{R}^k \times \bigoplus^k E^*) \rightarrow \mathbb{R}^k \times \bigoplus^k E^*$ given by

$$\tilde{\tau}_{\mathbb{R}^k \times \bigoplus^k E^*}^k(Z_{(\mathbf{t}, \mathbf{e}_q^*)}^1, \dots, Z_{(\mathbf{t}, \mathbf{e}_q^*)}^k) = (\mathbf{t}, \mathbf{e}_q^*),$$

where $Z_{(\mathbf{t}, \mathbf{e}_q^*)}^A = (a_{Aq}, v_{A(\mathbf{t}, \mathbf{e}_q^*)}) \in \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)$, $A = 1, \dots, k$. We have the following

Proposition 5.15. *Let $\xi = (\xi_1, \dots, \xi_k)$ be a section of $\tau_{\mathbb{R}^k \times (T_k^1)^*Q}^k$. Then*

$$(\rho^{\tilde{p}}(\xi_1), \dots, \rho^{\tilde{p}}(\xi_k)) : \mathbb{R}^k \times \bigoplus^k E^* \rightarrow T_k^1(\mathbb{R}^k \times \bigoplus^k E^*)$$

$$\xi_A : \mathbb{R}^k \times \bigoplus^k E^* \rightarrow \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)$$

is a k -vector field on $\mathbb{R}^k \times \bigoplus^k E^*$, where $\rho^{\tilde{p}}$ is the anchor map of the Lie algebroid $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)$.

Proof. Directly from (5.33) and the above remark. \blacksquare

5.2.4. *Hamiltonian formalism.* Let $(E, [\cdot, \cdot]_E, \rho_E)$ be a Lie algebroid on a manifold Q , and $H : \mathbb{R}^k \times \bigoplus^k E^* \rightarrow \mathbb{R}$ a Hamiltonian function. To develop the Hamiltonian k -cosymplectic formalism on Lie algebroids, we need to define an appropriate notion of Liouville sections.

The Liouville sections. The *Liouville 1-sections* are defined as sections of the bundle $(\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*))^* \rightarrow \mathbb{R}^k \times \bigoplus^k E^*$ such that

$$\begin{aligned} \Theta^A : \mathbb{R}^k \times \bigoplus^k E^* &\longrightarrow (\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*))^* & 1 \leq A \leq k, \\ (\mathbf{t}, \mathbf{e}_q^*) &\longmapsto \Theta_{(\mathbf{t}, \mathbf{e}_q^*)}^A \end{aligned}$$

where $\Theta_{(\mathbf{t}, \mathbf{e}_q^*)}^A : (\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*))_{(\mathbf{t}, \mathbf{e}_q^*)} \rightarrow \mathbb{R}$ is the linear function:

$$(5.36) \quad (a_q, v_{(\mathbf{t}, \mathbf{e}_q^*)}) \longmapsto \Theta_{(\mathbf{t}, \mathbf{e}_q^*)}^A(a_q, v_{(\mathbf{t}, \mathbf{e}_q^*)}) = e_{Aq}^*(a_q),$$

for each $a_q \in E$, $(\mathbf{t}, \mathbf{e}_q^*) = (\mathbf{t}, e_1^*, \dots, e_k^*) \in \mathbb{R}^k \times \bigoplus^k E^*$ and $v_{(\mathbf{t}, \mathbf{e}_q^*)} \in T_{(\mathbf{t}, \mathbf{e}_q^*)}(\mathbb{R}^k \times \bigoplus^k E^*)$. The *Liouville 2-sections*

$$\Omega^A : \mathbb{R}^k \times \bigoplus^k E^* \rightarrow \Lambda^2[\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)]^*, \quad 1 \leq A \leq k$$

defined by

$$\Omega^A = -d_{\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)} \Theta^A,$$

where $d_{\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)}$ denotes the exterior differential on the Lie algebroid $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)$ (see (5.35)).

Locally, if $\{\mathcal{Y}_B, \mathcal{X}_\alpha, \mathcal{V}_B^\beta\}$ is a local basis of $\text{Sec}(\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*))$ and $\{\mathcal{Y}^B, \mathcal{X}^\alpha, \mathcal{V}_\beta^B\}$ its dual basis, then from (5.32),

$$(5.37) \quad \Theta^A = \sum_{\beta=1}^m y_\beta^A \mathcal{X}^\beta, \quad 1 \leq A \leq k,$$

and from (5.34), (5.35) and (5.37),

$$(5.38) \quad \Omega^A = \sum_\beta \mathcal{X}^\beta \wedge \mathcal{V}_\beta^A + \frac{1}{2} \sum_{\beta, \gamma, \delta} y_\delta^A \mathcal{C}_{\beta\gamma}^\delta \mathcal{X}^\beta \wedge \mathcal{X}^\gamma, \quad 1 \leq A \leq k.$$

Remark 5.16. When $E = TQ$ and $\rho = id_{TQ}$ then

$$\Omega^A(X, Y) = \omega^A(X, Y), \quad 1 \leq A \leq k,$$

where X, Y are vector field on $\mathbb{R}^k \times (T_k^1)^*Q$ and $\omega^1, \dots, \omega^k$ are the canonical 2-forms of the standard Hamiltonian k -cosymplectic formalism (see (3.1)). \diamond

The Hamiltonian equations.

Theorem 5.17. Let $H : \mathbb{R}^k \times \bigoplus^k E^* \rightarrow \mathbb{R}$ be a Hamiltonian function and

$\xi_H = (\xi_1, \dots, \xi_k) : \mathbb{R}^k \times \bigoplus^k E \rightarrow (\mathcal{T}^E)_k^1(\mathbb{R}^k \times \bigoplus^k E^*) \cong \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*) \oplus \dots \oplus \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)$ a section of $\tilde{\tau}_{\mathbb{R}^k \times \bigoplus^k E^*}^k$, (or equivalently, ξ_1, \dots, ξ_k are k sections of the Hamiltonian prolongation), such that

$$(5.39) \quad \mathcal{Y}^B(\xi_A) = \delta_A^B, \quad \sum_{A=1}^k \iota_{\xi_A} \Omega^A = d^{\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)} H - \sum_{A=1}^k \frac{\partial H}{\partial t^A} \mathcal{Y}^A.$$

If $\psi : \mathbb{R}^k \rightarrow \mathbb{R}^k \times \bigoplus^k E^*$, $\psi(\mathbf{t}) = (\mathbf{t}, \psi^i(\mathbf{t}), \psi_\alpha^A(\mathbf{t}))$ is an integral section of ξ_H , then ψ is a solution of the following system of partial differential equations:

$$(5.40) \quad \begin{aligned} \frac{\partial \psi^i}{\partial t^A} \Big|_{\mathbf{t}} &= \rho_\alpha^i \frac{\partial H}{\partial y_\alpha^A} \Big|_{\psi(\mathbf{t})}, \\ \sum_{A=1}^k \frac{\partial \psi_\beta^A}{\partial t^A} \Big|_{\mathbf{t}} &= -\left(\rho_\beta^i \frac{\partial H}{\partial q^i} \Big|_{\psi(\mathbf{t})} + \sum_{A=1}^k \psi_\gamma^A(\mathbf{t}) \mathcal{C}_{\alpha\beta}^\gamma \frac{\partial H}{\partial y_\alpha^A} \Big|_{\psi(\mathbf{t})} \right). \end{aligned}$$

Remark 5.18. In the particular case $E = TQ$ and $\rho = id_{TQ}$, equations (5.40) are the Hamilton field equations. Accordingly, equations (5.40) are called the *Hamilton equations for Lie algebroids*. \diamond

Proof. The proof is analogous to that of Theorem 5.11 in section 5.1.5. A schedule of this proof is the following:

Consider $\{\mathcal{Y}_B, \mathcal{X}_\alpha, \mathcal{V}_B^\beta\}$, a local basis of sections of $\tilde{\tau}_{\mathbb{R}^k \times \bigoplus^k E^*}^k : \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*) \rightarrow \mathbb{R}^k \times \bigoplus^k E^*$. If $\xi_H = (\xi_1, \dots, \xi_k)$, then each component ξ_A can be written in the form

$$(5.41) \quad \xi_A = \xi_A^B \mathcal{Y}_B + \xi_A^\alpha \mathcal{X}_\alpha + (\xi_A)_\alpha^B \mathcal{V}_B^\alpha.$$

and from (5.35), (5.38) and (5.41) the local expression of (5.39) is

$$(5.42) \quad \xi_A^B = \delta_A^B, \quad \xi_A^\alpha = \frac{\partial H}{\partial y_\alpha^A}, \quad \sum_{A=1}^k (\xi_A)_\alpha^A = -\left(\rho_\alpha^i \frac{\partial H}{\partial q^i} + \sum_{A=1}^k \mathcal{C}_{\alpha\beta}^\gamma y_\gamma^A \frac{\partial H}{\partial y_\beta^A} \right).$$

Also, if $\psi : \mathbb{R}^k \rightarrow \mathbb{R}^k \times \bigoplus^k E^*$, $\psi(\mathbf{t}) = (\mathbf{t}, \psi^i(\mathbf{t}), \psi_\alpha^A(\mathbf{t}))$ is an integral section of ξ_H , that is ψ is an integral section of $(\rho^{\tilde{p}^*}(\xi_1), \dots, \rho^{\tilde{p}^*}(\xi_k))$, the associated k -vector field on $\mathbb{R}^k \times \bigoplus^k E^*$, then

$$(5.43) \quad \rho_\alpha^i (\xi_A^\alpha \circ \psi) = \frac{\partial \psi^i}{\partial t^A}, \quad (\xi_A)_\beta^B \circ \psi = \frac{\partial \psi_\beta^B}{\partial t^A}.$$

From (5.42) and (5.43),

$$\frac{\partial \psi^i}{\partial t^A} = \frac{\partial H}{\partial y_\alpha^A} \rho_\alpha^i \quad \text{and} \quad \sum_{A=1}^k \frac{\partial \psi_\beta^A}{\partial t^A} = -\left(\mathcal{C}_{\alpha\beta}^\gamma \psi_\gamma^A \frac{\partial H}{\partial y_\beta^A} + \rho_\alpha^i \frac{\partial H}{\partial q^i} \right).$$

Remark 5.19.

- (1) When H does not depends on t , then it can be considered as a map $H: \bigoplus^k E^* \rightarrow \mathbb{R}$. In this case, the sections Ω^A can be thought as sections of $\mathcal{T}^E(\bigoplus^k E^*)$ and from (5.39) one obtains the k -symplectic Hamiltonian equations for field theories on Lie algebroids (see [24]).
- (2) When $E = TQ$ and $\rho = id_{TQ}$, equations (5.39) are the standard k -cosymplectic geometric version of the Hamilton equations for field theories develop by M. de León *et al* in [30].
- (3) If H does not depends on t and we consider the case $E = TQ$ and $\rho = Id_{TQ}$ we obtain the standard k -symplectic geometric version of the Hamilton equations for field theories (see, for instance [43, 49]).

◊

In the following table we write the geometric Lagrangian equations in the above particular cases.

HAMILTONIAN FORMALISM Geometric Hamiltonian equations	
k -cosymplectic formalism on Lie algebroids	$\mathcal{Y}^B(\xi_A) = \delta_A^B$ $\sum_{A=1}^k \iota_{\xi_A} \Omega^A = d^{\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)} H - \sum_{A=1}^k \frac{\partial H}{\partial t^A} \mathcal{Y}^A$ (ξ_1, \dots, ξ_k) family of k sections of $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)$
k -symplectic formalism on Lie algebroids $(\frac{\partial H}{\partial t^A} = 0, A = 1, \dots, k)$	$\sum_{A=1}^k \iota_{\xi_A} \Omega^A = d^{\mathcal{T}^E(\bigoplus^k E^*)} H$ (ξ_1, \dots, ξ_k) family of k sections of $\mathcal{T}^E(\bigoplus^k E^*)$
Standard k -cosymplectic formalism ($E = TQ$)	$dt^A(Y_B) = \delta_B^A$ $\sum_{A=1}^k i_{Y_A} \omega^A = dH - \sum_{A=1}^k \frac{\partial H}{\partial t^A} dt^A$ (Y_1, \dots, Y_k) k -vector field on $\mathbb{R}^k \times (T_k^1)^* Q$
Standard k -symplectic formalism ($E = TQ$) $(\frac{\partial H}{\partial t^A} = 0, A = 1, \dots, k)$	$\sum_{A=1}^k i_{Y_A} \omega_L^A = dH$ (Y_1, \dots, Y_k) k -vector field on $(T_k^1)^* Q$

Remark 5.20. When $k = 1$,

- (1) If H explicitly depends on t , equations (5.39) are the equations of Hamiltonian mechanics for time-dependent system defined on Lie algebroids (see [51, 52], for instance). Moreover, when $E = TQ$ and $\rho = Id_{TQ}$ we have the dynamical equations of the non-autonomous mechanics (see [11]).
- (2) If H does not depends on t , equations (5.39) are the geometric equations of autonomous Hamiltonian mechanics on Lie algebroids (see, for instance, [38]). In this case, if $E = TQ$ and $\rho = id_{TQ}$ we have the classical equations of autonomous mechanics.

◊

5.3. Equivalence between the Lagrangian and Hamiltonian formalism. In the standard case the Hamiltonian and Lagrangian k -cosymplectic formulations are equivalents when the Lagrangian is hyperregular. On the k -symplectic formalism on Lie algebroid we have obtained a similar result (see [24]). In this section we will define the Legendre transformation on Lie algebroids and we will establish the equivalence between the Lagrangian and Hamiltonian formalisms when the Lagrangian function is hyperregular.

Definition 5.21. *The Legendre transformation associated with $L : \mathbb{R}^k \times \bigoplus^k E \rightarrow \mathbb{R}$ is the smooth map*

$$Leg : \mathbb{R}^k \times \bigoplus^k E \rightarrow \mathbb{R}^k \times \bigoplus^k E^*$$

defined by

$$Leg(\mathbf{t}, \mathbf{e}_q) = \left(\mathbf{t}, [Leg(\mathbf{t}, \mathbf{e}_q)]^1, \dots, [Leg(\mathbf{t}, \mathbf{e}_q)]^k \right)$$

where

$$[Leg(\mathbf{t}, \mathbf{e}_q)]^A(u_q) = \frac{d}{ds} \Big|_{s=0} L(\mathbf{t}, e_{1_q}, \dots, e_{A_q} + su_q, \dots, e_{k_q}), \quad 1 \leq A \leq k,$$

where $u_q \in E_q$ and $(\mathbf{t}, \mathbf{e}_q) = (\mathbf{t}, e_{1_q}, \dots, e_{k_q}) \in \mathbb{R}^k \times \bigoplus^k E$.

The map Leg is well defined, and its local expression is

$$Leg(t^A, q^i, y_A^\alpha) = (t^A, q^i, \frac{\partial L}{\partial y_A^\alpha}).$$

From this expression, it is easy to prove that the Lagrangian L is regular if and only if Leg is a local diffeomorphism.

Remark 5.22. When $E = TQ$, the Legendre transformation defined here coincides with the Legendre map of the standard k -cosymplectic formalism, see [31, 44]. \diamond

Leg induces a map

$$\mathcal{T}^E Leg : \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E) \rightarrow \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)$$

defined by

$$\mathcal{T}^E Leg(a_q, v_{(\mathbf{t}, \mathbf{e}_q)}) = \left(a_q, (Leg)_*(\mathbf{t}, \mathbf{e}_q)(v_{(\mathbf{t}, \mathbf{e}_q)}) \right),$$

where $a_q \in E_q$, $(\mathbf{t}, \mathbf{e}_q) \in \mathbb{R}^k \times \bigoplus^k E$ and $(a_q, v_{(\mathbf{t}, \mathbf{e}_q)}) \in \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E) \subset E \times T(\mathbb{R}^k \times \bigoplus^k E)$.

Theorem 5.23. *The pair $(\mathcal{T}^E Leg, Leg)$ is a morphism between the Lie algebroid $(\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E), \rho^{\tilde{p}}, [\cdot, \cdot]^{\tilde{p}})$ and $(\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*), \rho^{\tilde{p}^*}, [\cdot, \cdot]^{\tilde{p}^*})$.*

$$\begin{array}{ccc} \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E) & \xrightarrow{\mathcal{T}^E Leg} & \mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*) \\ \tilde{\tau}_{\mathbb{R}^k \times \bigoplus^k E} \downarrow & & \downarrow \tilde{\tau}_{\mathbb{R}^k \times \bigoplus^k E^*} \\ \mathbb{R}^k \times \bigoplus^k E & \xrightarrow{Leg} & \mathbb{R}^k \times \bigoplus^k E^* \end{array}$$

Moreover, if Θ_L^A and Ω_L^A are, respectively, the Poincaré-Cartan 1-sections and 2-sections associated with $L : \mathbb{R}^k \times \bigoplus^k E \rightarrow \mathbb{R}$, and Θ^A and Ω^A , respectively, the Liouville 1-sections and 2-sections on $\mathcal{T}^E(\mathbb{R}^k \times \bigoplus^k E^*)$, then

$$(5.44) \quad (\mathcal{T}^E Leg, Leg)^* \Theta^A = \Theta_L^A, \quad (\mathcal{T}^E Leg, Leg)^* \Omega^A = \Omega_L^A, \quad 1 \leq A \leq k.$$

Proof. The proof is analogous to the one in the k -symplectic case, see Theorem 4.30 in [24]. \blacksquare

Remark 5.24. When $E = TQ$ and $\rho = id_{TQ}$, it establishes the relation between the Lagrangian and Hamiltonian formalism in the standard k -cosymplectic approach (see [31]). \diamond

We next assume that L is *hyperregular*, that is, that Leg is a global diffeomorphism. In this case we may consider the Hamiltonian function $H: \mathbb{R}^k \times \bigoplus^k E^* \rightarrow \mathbb{R}$ defined by

$$H = E_L \circ (\text{Leg})^{-1},$$

where E_L is the energy function associated with L , given by (5.24), and $(\text{Leg})^{-1}$ is the inverse of the Legendre transformation.

$$\begin{array}{ccc} \mathbb{R}^k \times \bigoplus^k E^* & \xrightarrow{\text{Leg}^{-1}} & \mathbb{R}^k \times \bigoplus^k E \\ & \searrow H & \downarrow E_L \\ & & \mathbb{R} \end{array}$$

By a similar computation that in the Theorem 4.33 in [24] we prove the following theorem, which establishes the equivalence between the Lagrangian and Hamiltonian k -cosymplectic formulations on Lie algebroids.

Theorem 5.25. *Let L be a hyperregular Lagrangian. There is a bijective correspondence between the set of maps $\eta: \mathbb{R}^k \rightarrow \mathbb{R}^k \times \bigoplus^k E$ such that η is an integral section of a solution ξ_L of the geometric Euler-Lagrange equations (5.26) and the set of maps $\psi: \mathbb{R}^k \rightarrow \mathbb{R}^k \times \bigoplus^k E^*$ which are integral sections of some solution ξ_H of the geometric Hamilton equations (5.40).*

Proof. It is similar to the proof of the k -symplectic formalism on Lie algebroids, see Theorem 4.33 in [24]. Here we must only to take into account the relationship between $\xi_L = (\xi_L^1, \dots, \xi_L^k)$ and $\xi_H = (\xi_H^1, \dots, \xi_H^k)$ given by:

$$\xi_H^A \circ \text{Leg} = \mathcal{T}^E \text{Leg} \circ \xi_L^A, \quad A = 1, \dots, k.$$

■

Remark 5.26.

- (1) When $E = TQ$, this theorem establishes the equivalence between the k -cosymplectic Lagrangian and the Hamiltonian formalism (see [31, 44]).
- (2) When L and H do not depend on \mathbf{t} , the above Theorem reduces to the Theorem 4.33 in [24], which establishes the equivalence between the Lagrangian and Hamiltonian formalism on Lie algebroids on the k -symplectic approach.

◊

6. EXAMPLES

Harmonic maps, [5, 12]. Let us remember that a smooth map $\varphi: M \rightarrow N$ between Riemannian manifolds (M, g) and (N, h) is called *harmonic* if it is a critical point of the energy functional E , which, when M is compact oriented manifold, is defined as

$$E(\varphi) = \int_M \frac{1}{2} \text{trace}_g \varphi^* h \, dv_g$$

where dv_g denotes the measure on M induced by its metric and, in local coordinates, the expression $\frac{1}{2} \text{trace}_g \varphi^* h$ reads

$$\frac{1}{2} g^{ij} h_{\alpha\beta} \frac{\partial \varphi^\alpha}{\partial x^i} \frac{\partial \varphi^\beta}{\partial x^j}.$$

This definition is extended to the case when M is not compact requiring that the restriction of φ to every compact domain to be harmonic.

Now we will consider the particular case $M = \mathbb{R}^k$ and $N = G$ a Riemannian matrix Lie group. In this case we denote the trivial principal fiber bundle by $\pi: \mathbb{R}^k \times G \rightarrow \mathbb{R}^k$, and we identify $\text{Sec}(\mathbb{R}^k \times G)$ with $\mathcal{C}^\infty(\mathbb{R}^k, G)$. For each $\phi \in \mathcal{C}^\infty(\mathbb{R}^k, G)$, the Riemannian metrics on \mathbb{R}^k and G naturally induce a metric $\langle \cdot, \cdot \rangle$ on $\mathcal{C}^\infty(T^* \mathbb{R}^k \otimes \varphi^*(TG))$, and so we may define the energy E on $\mathcal{C}^\infty(\mathbb{R}^k, M)$ by

$$(6.1) \quad E(\varphi) = \int_{\mathbb{R}^k} L(\varphi^{[1]}(t)) d^k t$$

where $L(\varphi^{[1]}(t)) = \frac{1}{2} \langle T\varphi, T\varphi \rangle$ and $d^k t = dt^1 \wedge \dots \wedge dt^k$ is the volume element of \mathbb{R}^k .

The Euler-Lagrange equations for (6.1) are given by (see, for example [12]),

$$(6.2) \quad \text{Trace}(\nabla T\phi) = 0,$$

where ∇ is the induced Riemannian covariant derivative on $\mathcal{C}^\infty(T^*\mathbb{R}^k \otimes \varphi^*(TG))$ and Trace is the trace defined by g (see, for example [12]). By definition, the set of harmonic maps from \mathbb{R}^k to G is the subset of $\text{Sec}(\mathbb{R}^k \times G)$ whose elements solve (6.2).

Using Einstein's summation convention, we have the following coordinate expressions:

$$(6.3) \quad \begin{aligned} L: \quad J^1(\mathbb{R}^k \times G) \equiv \mathbb{R}^k \times T_k^1 G &\rightarrow \mathbb{R} \\ (t^A, q^i, v_A^i) &\mapsto L(t^A, q^i, v_A^i) = \frac{1}{2}g^{AB}v_A^i v_B^j h_{ij} \end{aligned}$$

where t^A denoted the local coordinates on \mathbb{R}^k , that is, the space-time coordinates, $q^i = \varphi^i$ the components of the field φ and $v_A^i = \frac{\partial \varphi^i}{\partial t^A}$ the partial derivatives of the components of the field. From (6.2) one obtains

$$(6.4) \quad g^{AB} \left(\frac{\partial^2 \varphi^i}{\partial t^A \partial t^B} - \Gamma_{AB}^C \frac{\partial \varphi^i}{\partial t^C} + \tilde{\Gamma}_{jk}^i \frac{\partial \varphi^j}{\partial t^A} \frac{\partial \varphi^k}{\partial t^B} \right) = 0 \quad 1 \leq i \leq n,$$

where Γ_{AB}^C and $\tilde{\Gamma}_{jk}^i$ denote the Christoffel symbols of the Levi-Civita connections of g and h .

We shall derive the reduced form of (6.2) for two specific cases: $G = \mathbb{R}$ and $G = \mathbb{S}^3 \cong SU(2)$. In general, one obtains,

$$(6.5) \quad \mathcal{C}(\mathbb{R}^k \times G) \cong (J^1(\mathbb{R}^k \times G)/G \cong (\mathbb{R}^k \times T_k^1 G)/G \cong (\mathbb{R}^k \times G \times \mathfrak{g} \times \dots \times \mathfrak{g})/G \cong \mathbb{R}^k \times \mathfrak{g} \times \dots \times \mathfrak{g},$$

where $\mathcal{C}(\mathbb{R}^k \times G) \rightarrow \mathbb{R}^k$ is the bundle of connections (see [7]).

For the case that $G = \mathbb{R}$, the abelian group of translations, from (6.5) we obtain that $\mathcal{C}(\mathbb{R}^k \times G) \cong \mathbb{R}^k \times \mathbb{R}^k$ and therefore, a section σ of the bundle connections can be thought as a 1-form on \mathbb{R}^k with local expression $\sigma = p_A dt^A$, where (t^A, p_B) are local coordinates on $\mathbb{R}^k \times \mathbb{R}^k$.

The Lagrangian L is clearly \mathbb{R} -invariant. Denoting by ℓ the projection of L to $\mathcal{C}(P) \cong \mathcal{C}(\mathbb{R}^k \times G) \cong \mathbb{R}^k \times \mathbb{R}^k$, in local coordinates, we obtain $\ell(t^A, p_B) = \frac{1}{2}g^{AB}p_A p_B$. Now, we can write the Euler-Lagrange equations (5.27) for this Lagrangian ℓ and we obtain

$$(6.6) \quad \begin{aligned} \frac{\partial(g^{AB}p_B)}{\partial t^A} + \Gamma_{AC}^A g^{CB} p_B &= 0 \\ \frac{\partial p_A}{\partial t^B} - \frac{\partial p_B}{\partial t^A} &= 0. \end{aligned}$$

Let us observe that the first equation is the Euler-Poincaré equation for ℓ and the second equation is the condition of the vanishing curvature on the trivial connection for $\mathbb{R}^k \times G$ (see, for instance, [7]).

We denote by $q: \mathbb{R}^k \times T_k^1 \mathbb{R} \rightarrow (\mathbb{R}^k \times T_k^1 \mathbb{R})/\mathbb{R}$ the canonical projection, let $\sigma = q(T\varphi)$, then $p_A = \partial\varphi/\partial t^A$, this condition together the equations (6.6) is equivalent to (6.4).

For the case $G = \mathbb{S}^3 \cong SU(2)$, from (6.5) we know that $\mathcal{C}(\mathbb{R}^k \times SU(2)) \cong \mathbb{R}^k \times \mathfrak{su}(2) \times \dots \times \mathfrak{su}(2)$ and we can make the identification

$$T^*\mathbb{R}^k \otimes \mathfrak{su}(2) \cong \mathbb{R}^k \times \mathfrak{su}(2) \times \dots \times \mathfrak{su}(2).$$

This identification is locally given as follow: Let $\{E_1, E_2, E_3\}$ be a basis of $\mathfrak{su}(2)$, then a section of $T^*\mathbb{R}^k \otimes \mathfrak{su}(2) \rightarrow \mathbb{R}^k$ can be written as $\sigma(t) = p_i^A dt^A \otimes E_i$. This element σ identifies with the element of $\mathbb{R}^k \times \mathfrak{su}(2) \times \dots \times \mathfrak{su}(2)$ with local coordinates (t^A, p_i^A) .

The lagrangian L , (see (6.3)), is $\mathfrak{su}(2)$ -invariant and its projection to $\mathbb{R}^k \times \mathfrak{su}(2) \times \dots \times \mathfrak{su}(2)$ is

$$\ell(t^A, p_i^A) = \frac{1}{2}g^{AB}p_i^A p_j^B h_{ij}.$$

Then the Euler-Lagrange equations (5.27) write, in this case, as follow:

$$(6.7) \quad \begin{aligned} \frac{\partial(g^{AB}p_i^B h_{ij})}{\partial t^A} + \Gamma_{CB}^C g^{AB} p_i^A h_{ij} + g^{AB} p_i^A p_k^B c_{kj}^l h_{il} &= 0 \\ \frac{\partial p_k^A}{\partial t^B} - \frac{\partial p_k^B}{\partial t^A} + p_i^A p_j^B c_{ij}^k &= 0. \end{aligned}$$

The first group of equations of (6.7) are the Euler-Poincaré equations for the trivial connection of $\mathbb{R}^k \times SU(2)$, the second group of equations represents the vanishing curvature condition (see [7] for more details).

Classical Euler-Poincaré equations. For a Lie Group G , we consider the principal fiber bundle $\pi: \mathbb{R} \times G \rightarrow \mathbb{R}$. Let $L: J^1(\mathbb{R} \times G) \cong \mathbb{R} \times TG \rightarrow \mathbb{R}$ be a G -invariant Lagrangian. Taking into account (6.5) with $k = 1$ we obtain the following identifications

$$\mathcal{C}(\mathbb{R} \times G) \cong (\mathbb{R} \times TG)/G \cong \mathbb{R} \times \mathfrak{g}.$$

In a similar way that in the above example we obtain if ℓ is the projection of L to $\mathcal{C}(\mathbb{R} \times G)$, then the Euler-Lagrange equations associated to ℓ are the Classical Euler-Poincaré equations, see for instance [7] or [36].

Systems with symmetry. Consider a principal bundle $\pi: \bar{Q} \rightarrow Q = \bar{Q}/G$. Let $A: T\bar{Q} \rightarrow \mathfrak{g}$ be a fixed principal connection with curvature $B: T\bar{Q} \oplus T\bar{Q} \rightarrow \mathfrak{g}$. The connection A determines an isomorphism between the vector bundles $T\bar{Q}/G \rightarrow Q$ and $TQ \oplus \tilde{\mathfrak{g}} \rightarrow Q$, where $\tilde{\mathfrak{g}} = (\bar{Q} \times \mathfrak{g})/G$ is the adjoint bundle (see [8]):

$$[v_{\bar{q}}] \leftrightarrow T_{\bar{q}}\pi(v_{\bar{q}}) \oplus [(\bar{q}, A(v_{\bar{q}}))]$$

where $v_{\bar{q}} \in T_{\bar{q}}\bar{Q}$. The connection allows us to obtain a local basis of sections of $\text{Sec}(T\bar{Q}/G) = \mathfrak{X}(Q) \oplus \text{Sec}(\tilde{\mathfrak{g}})$ as follows. Let \mathfrak{e} be the identity element of the Lie group G and assume that there are local coordinates (q^i) , $1 \leq i \leq \dim Q$ and that $\{\xi_a\}$ is a basis of \mathfrak{g} . The corresponding sections of the adjoint bundle are the left-invariant vector fields ξ_a^L :

$$\xi_a^L(g) = T_{\mathfrak{e}}L_g(\xi_a)$$

where $L_g: G \rightarrow G$ is left translation by $g \in G$. If

$$A\left(\frac{\partial}{\partial q^i}_{(q,\mathfrak{e})}\right) = A_i^a \xi_a$$

then the corresponding horizontal lifts on the trivialization $U \times G$ are the vector fields

$$\left(\frac{\partial}{\partial q^i}\right)^h = \frac{\partial}{\partial q^i} - A_i^a \xi_a^L.$$

The elements of the set

$$\left\{ \left(\frac{\partial}{\partial q^i}\right)^h, \xi_a^L \right\}$$

are by construction G -invariant, and therefore, constitute a local basis of sections $\{e_i, e_a\}$ of $\text{Sec}(T\bar{Q}/G) = \mathfrak{X}(Q) \oplus \text{Sec}(\tilde{\mathfrak{g}})$.

Denote by (q^i, y^i, y^a) the induced local coordinates of $T\bar{Q}/G$. Then

$$B\left(\frac{\partial}{\partial q^i}_{(q,\mathfrak{e})}, \frac{\partial}{\partial q^j}_{(q,\mathfrak{e})}\right) = B_{ij}^a \xi_a$$

where

$$B_{ij}^c = \frac{\partial A_i^c}{\partial q^j} - \frac{\partial A_j^c}{\partial q^i} - \mathcal{C}_{ab}^c A_i^a A_j^b,$$

the \mathcal{C}_{ab}^c being the structure constants of the Lie algebra. The structure functions of the Lie algebroid $T\bar{Q}/G \rightarrow Q$ are determined (see [25]) by

$$\begin{aligned} [e_i, e_j]_{T\bar{Q}/G} &= -B_{ij}^c e_c \\ [e_i, e_a]_{T\bar{Q}/G} &= \mathcal{C}_{ab}^c A_i^b e_c \\ [e_a, e_b]_{T\bar{Q}/G} &= \mathcal{C}_{ab}^c e_c \\ \rho_{T\bar{Q}/G}(e_i) &= \frac{\partial}{\partial q^i} \\ \rho_{T\bar{Q}/G}(e_a) &= 0, \end{aligned}$$

and for a Lagrangian function $L : \mathbb{R}^k \times \bigoplus^k T\bar{Q}/G \rightarrow \mathbb{R}$ the Euler-Lagrange field equations are

$$\begin{aligned} \frac{d}{dt^A} \left(\frac{\partial L}{\partial y_A^i} \right) &= \frac{\partial L}{\partial q^i} + B_{ij}^c y_C^j \frac{\partial L}{\partial y_C^c} - \mathcal{C}_{ab}^c A_i^b y_C^a \frac{\partial L}{\partial y_C^c} \\ \frac{d}{dt^A} \left(\frac{\partial L}{\partial y_A^a} \right) &= \mathcal{C}_{ab}^c A_i^b y_C^i \frac{\partial L}{\partial y_C^c} - \mathcal{C}_{ab}^c y_C^b \frac{\partial L}{\partial y_C^c} \\ 0 &= \frac{\partial y_A^i}{\partial t^B} - \frac{\partial y_B^i}{\partial t^A} \\ 0 &= \frac{\partial y_A^c}{\partial t^B} - \frac{\partial y_B^c}{\partial t^A} - B_{ij}^c y_B^i y_A^j + \mathcal{C}_{ab}^c A_i^b y_B^a y_A^a + \mathcal{C}_{ab}^c y_A^b y_B^a. \end{aligned}$$

If Q is a single point, that is, $\bar{Q} = G$, then $T\bar{Q}/G = \mathfrak{g}$, the Lagrangian is a function $L : \mathbb{R}^k \times \bigoplus^k \mathfrak{g} \rightarrow \mathbb{R}$, and the field equations reduce to

$$\begin{aligned} \frac{d}{dt^A} \left(\frac{\partial L}{\partial y_A^a} \right) &= -c_{ab}^c y_C^b \frac{\partial L}{\partial y_C^c} \\ 0 &= \frac{\partial y_A^c}{\partial t^B} - \frac{\partial y_B^c}{\partial t^A} + \mathcal{C}_{ab}^c y_A^b y_B^a \end{aligned}$$

a local form of the Euler-Poincaré equations in field theory (see, for instance, [5] and [39]).

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